Annealed tail estimates for a Brownian motion in a drifted Brownian potential

Marina Talet

Abstract

We study Brownian motion in a drifted Brownian potential. Kawazu and Tanaka [23] exhibited two speed regimes for this process, depending on the drift. They supplemented these laws of large numbers by central limit theorems, which were recently completed by Hu, Shi and Yor [19] using stochastic calculus. We studied large deviations [34], showing among other results that the rate function in the annealed setting, that is after averaging over the potential, has a flat piece in the ballistic regime. In this paper, we focus on this subexponential regime, proving that the probability of deviating below the almost sure speed has a polynomial rate of decay, and computing the exponent in this power law. This provides the continuous-time analogue of what Dembo, Peres and Zeitouni proved for the transient random walk in random environment [13]. Our method takes a completely different route, making use of Lamperti's representation together with an iteration scheme.

1 Introduction

Let $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ be a collection of i.i.d. $(0,1)$-valued random variables, serving as an environment, and define a conditional Markov chain on the integers, $\{S_n\}_{n\geq 0}$, by $S_0 = 0$ and

$$\mathbb{P}\left(S_{n+1} = y \,|\, S_n = x,\ \{\omega_i\}_{i\in\mathbb{Z}}\right) = \begin{cases} \omega_x & \text{if } y = x+1, \\ 1-\omega_x & \text{if } y = x-1, \\ 0 & \text{otherwise.} \end{cases}$$

The process $\{S_n\}_{n\geq 0}$ is called a random walk in a random environment (hereafter abbreviated RWRE).

Solomon [33] completely solved the transience/recurrence problem for $\{S_n\}_{n\geq 0}$, and determined furthermore the speed of the walk. In particular, setting $\rho \stackrel{\text{def}}{=} (1-\omega_0)/\omega_0$, he proved that if $\mathbb{E}(\rho) < 1$, then, almost surely, $\lim_{n\to\infty} S_n/n = (1-\mathbb{E}\rho)/(1+\mathbb{E}\rho) \stackrel{def}{=} \mathrm{v}$ (with the case $\mathbb{E}(1/\rho) < 1$ then following by reflection), and that otherwise $\lim_{n\to\infty} S_n/n = 0$ almost surely. These laws of large numbers were later developed into central limit theorems by Kesten, Kozlov and Spitzer [25].

Large deviations for $\{S_n/n\}_{n>0}$ were investigated by several authors, both under the conditional probability given ω, the so-called *quenched* probability P^ω, and the *annealed* one $\mathbb{P}$, that given after

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averaging over the environment ω. We refer to Greven and den Hollander [18], Gantert and Zeitouni [16] and Comets, Gantert and Zeitouni [9] for insightful overviews.

Dembo, Peres and Zeitouni [13] studied the probability with which the walk deviates from its limiting speed when this speed is nonzero. Assuming $\mathbb{E}(\rho) < 1$ and $\mathbb{P}(\omega_0 < 1/2) > 0$, let us write s for the unique $s > 1$ such that

$$\mathbb{E}(\rho^s) = 1. \tag{1.1}$$

They proved that

Theorem A ([13]) *For any open $G \subset (0,\mathrm{v})$ which is separated from* v,

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left(\frac{S_n}{n} \in G\right) = 1 - s.$$

This result was the starting point for our work.

In the present paper, we are interested in the continuous-time analogue of RWREs, the so-called Brownian motion in random potential $W = \{W(x)\}_{x\in\mathbb{R}}$. This should be a solution of the *formal* stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\alpha(t) - \frac{1}{2}\, W'(X(t))\,\mathrm{d}t, \\ X(0) = 0, \end{cases}$$

where the potential W is defined by

$$W(x) = B(x) - \frac{\kappa}{2} x, \qquad x \in \mathbb{R}, \qquad \kappa \in \mathbb{R},$$

with $\{B(x)\}_{x\in\mathbb{R}}$ is a one-dimensional two-sided Brownian motion defined on $\mathbb{R}$ starting from zero and $\{\alpha(x)\}_{x\geq 0}$ a standard Brownian motion such that $\alpha(0) = 0$, independent of W (or equivalently of B).

One way of defining a "formal solution" is this: since the Brownian motion is almost surely nowhere differentiable, one defines the process X through its conditional generator given W,

$$\mathcal{L}_W \stackrel{\text{def}}{=} \frac{1}{2}\, \mathrm{e}^{W(x)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{-W(x)} \frac{\mathrm{d}}{\mathrm{d}x} \right).$$

Since we are dealing with one-dimensional diffusions, there is a second approach to defining X, which we shall adopt. The martingale representation for diffusions tells us that, at fixed environment, that is to say for each realization of the environment W, the image of X under its scale function, which is a continuous martingale, can be represented as a time-changed Brownian motion. Namely,

$$X(t) = S^{-1}(\mathcal{B}(T^{-1}(t))), \qquad t \geq 0, \tag{1.2}$$

where $\{\mathcal{B}(t)\}_{t\geq 0}$ is a standard Brownian motion starting from 0, independent of W, with the scale function S and random clock T defined by

$$S(x) \stackrel{\text{def}}{=} \int_0^x \mathrm{e}^{W(y)}\,\mathrm{d}y, \qquad x \in \mathbb{R}, \tag{1.3}$$

$$T(t) \stackrel{\text{def}}{=} \int_0^t \exp\left(-2W(S^{-1}(\mathcal{B}(u)))\right)\,\mathrm{d}u, \qquad t \geq 0, \tag{1.4}$$

where S^{-1} and T^{-1} denote the respective inverse functions of S and T.

In the *quenched* situation, i.e. at fixed environment, X is Markov. We denote its law by P^W and the Wiener measure by Q. Averaging P^W over Q gives birth to a new probability $\mathbb{P}$, called the *annealed* probability. Note that, under $\mathbb{P}$, the process X is not necessarily Markov.

Brox [2] was the first to study such processes. He proved that for $\kappa = 0$, in which case the diffusion is recurrent, the motion is extremely slow, as then $X(t)$ is of order $\log^2 t$ for large t, in this way differing markedly from the diffusive behavior of Brownian motion.

For $\kappa \neq 0$, X is transient to the left or the right depending on the sign of κ, which is just "space reversal invariance". Kawazu and Tanaka [23] computed its almost-sure speed; assuming $\kappa > 0$,

$$\lim_{t\to\infty} \frac{X(t)}{t} = v_\kappa \stackrel{\text{def}}{=} \frac{(\kappa-1)^+}{4}, \qquad \mathbb{P}-\text{a.s.}$$

These are continuous-time analogues of Solomon's aforementioned laws of large numbers for RWRE. The corresponding central limit theorems were established by Kawazu and Tanaka [24] using Krein's spectral theory, and both recovered and completed by Hu, Shi and Yor [19] using stochastic calculus. We proved in [34] that the family of distributions of $\{X(t)/t\}_{t>0}$ satisfies a Large Deviation Principle in both the quenched and the annealed frameworks. We note that κ plays the role of s (defined for the RWRE in (1.1)) for these laws of large numbers, for central limit theorems as well as for the results of the present paper.

As in the discrete case, we are interested in the probability with which X deviates from its limiting speed v_κ when $v_\kappa \neq 0$, in the annealed setting. By symmetry, we only have to deal with $\kappa > 1$ in which case, according to [34], the annealed rate function has a "flat" piece, by which one means that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{X(t)}{t} \in G\right) = 0,$$

for any open set $G \subset (0, v_\kappa)$ which is separated from v_κ. This tells us that the probability for $X(t)/t$ to deviate below the typical velocity decays subexponentially fast to zero.

A natural question arises : how fast exactly? And can a result similar to Theorem A be obtained for X? The answer is provided by:

Theorem 1.1 *Let $\kappa > 1$. For any open $G \subset (0, v_\kappa)$ which is separated from v_κ,*

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}\left(\frac{X(t)}{t} \in G\right) = 1 - \kappa.$$

Again, as in [34], following the approach of [9], we set

$$H(r) \stackrel{\text{def}}{=} \inf\{t > 0 : \; X(t) > r\}, \qquad r > 0. \tag{1.5}$$

Making use of the "natural duality" between the diffusion X and its first hitting time process H, that is to say

$$\mathbb{P}\left(\frac{X(t)}{t} \sim \; v\right) \; \approx \; \mathbb{P}\left(\frac{H(tv)}{t} \; \sim \frac{1}{v}\right),$$

the proof of Theorem 1.1 reduces to showing

Theorem 1.2 *Let $\kappa > 1$. For any $u > v_\kappa^{-1} = 4/(\kappa-1)$,*

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}(H(r) > ur) \le 1 - \kappa, \tag{1.6}$$

and for any $u > 0$,

$$\liminf_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}(H(r) > ur) \ge 1 - \kappa. \tag{1.7}$$

Differently from [13], our proof hinges upon stochastic calculus techniques. A key role is played by Lamperti's representation which relates the potential W to a Bessel process. This fact enabled Hu, Shi and Yor to derive central limit theorems for the model we are studying, in [19]. Here we are interested in deviation estimates, hence in the rate at which various random variables involved in [19] converge. This leads to the delicate probability estimates of Sections 5 and 6.

We note from a glance at both [24] and [19] that in the case where $1 < \kappa < 2$,

$$\frac{1}{r^{1/\kappa}} \left(H(r) - \frac{4}{\kappa - 1}\, r\right) \stackrel{law}{\longrightarrow} \; \text{stable variable,}$$

where $\stackrel{law}{\longrightarrow}$ denotes convergence in distribution. In this case, as we proved in [36], the main contribution to the polynomial rate of decay of $\mathbb{P}(H(r) > ur)$, ($\sim r^{1-\kappa}$), stems from the limiting stable law in this regime.

To prove Theorem 1.2, we introduce an iteration technique which (as far as we know) is new and may prove to be of use elsewhere.

Solely using integration by parts followed by a time change (see (5.18) and (6.6)), together with results on Bessel and stable processes, our iteration blends very naturally with the techniques used in this paper. It also offers an alternative strategy that circumvents dealing with special functions while solving a Sturm-Liouville equation; see Section 8.

The outline of the paper is as follows. In Section 2, we list a collection of known results on Brownian local times, Bessel and Jacobi processes. In Section 3, we state and prove three lemmas which will be of frequent use in the proof of our tail estimates. The main result of this paper is Theorem 1.1. As in the discrete case, proving Theorem 1.1 reduces to proving Theorem 1.2 for the first hitting time process. This step is justified in Section 7.2, and the rest of the paper is devoted to the proof

of Theorem 1.2. This proceeds in one further step: Theorem 4.1, stated in Section 4. We prove Theorem 4.1 in Section 6 and Section 7.1 by means of a key estimate stated and proved in Section 5. Theorem 4.1 implies Theorem 1.2; this is proved in Section 4. And the last section is devoted to solving a Sturm-Liouville equation, providing an alternative method to our iteration technique.

Notation: Throughout the sequel, Q will denote the Wiener measure, E_Q the expectation with respect to Q, P_x^W and $\mathbb{P}_x \stackrel{def}{=} E_Q(P_x^W(.))$ the quenched and annealed laws when $X(0) = x$, and E_x^W and $\mathbb{E}_x$ the expectations w.r.t. P_x^W and $\mathbb{P}_x$ respectively. For notational convenience, P^W, $\mathbb{P}$, E^W and $\mathbb{E}$ stand for P_0^W, $\mathbb{P}_0$, E_0^W and $\mathbb{E}_0$. We sometimes drop the subscript x in $\mathbb{P}_x$ or in $\mathbb{E}_x$, for $x \neq 0$, when no confusion can arise.

Unless stated otherwise, **it is assumed that** $\kappa > 1$.

2 Preliminaries

In this section, we summarize a collection of known results which will be useful in the rest of the paper. These results concern Brownian local times, Bessel and Jacobi processes as well as Lamperti's representation for exponentials of drifted Brownian motions.

Let $\{\gamma(t)\}_{t\geq 0}$ be a standard Brownian motion. A well-known theorem of Trotter [37] confirms the existence of a jointly continuous version of the local time process $\{L_t^x(\gamma)\}_{t\geq 0,\, x\in\mathbb{R}}$ as the density of occupation time: for any bounded Borel function f,

$$\int_0^t f(\gamma(s))\,\mathrm{d}s = \int_{\mathbb{R}} f(x) L_t^x(\gamma)\,\mathrm{d}x. \tag{2.1}$$

Let

$$\sigma_\gamma(r) \stackrel{\text{def}}{=} \inf\{t > 0 :\ \gamma(t) > r\}, \qquad r > 0, \tag{2.2}$$

$$\tau_\gamma(r) \stackrel{\text{def}}{=} \inf\{t > 0 :\ L_t^0(\gamma) > r\}, \qquad r > 0, \tag{2.3}$$

denote the first hitting time of γ and its inverse local time at 0 respectively. We shall drop the subscript γ when no confusion arises.

As is shown by the Ray–Knight theorems (see Revuz and Yor [31], Chap XI), Brownian local times at these hitting times are nice diffusion processes, known as Bessel processes.

Definition. *A squared Bessel process $\{R^2(t), t \geq 0, \mathbb{P}\}$ of dimension d and started at r^2, is the solution of the stochastic differential equation*

$$\mathrm{d}R^2(t) = 2R(t)\,\mathrm{d}\gamma(t) + d\ \mathrm{d}t, \tag{2.4}$$

with $R^2(0) = r^2$ and γ a $\mathbb{P}$-Brownian motion. A Bessel process of dimension d, started at r, is $\{R(t), t \geq 0, \mathbb{P}\}$ with $R(t) \geq 0$ and $R(0) = r$.

We recall seven facts from the literature.

Fact 1 *(Ray-Knight theorems)*

First: *The process* $\{L^{1-t}_{\sigma(1)}\}_{t\geq 0}$ *is a squared Bessel process started at 0, of dimension 2 for* $0 \leq t \leq 1$ *and of dimension 0 for* $t \geq 1$. *(Here* $L^0_{\sigma(1)}$ *is an exponential random variable of mean 2.)*

Second: *The process* $\{L^t_{\tau(1)}\}_{t\geq 0}$ *is a squared Bessel process of dimension 0, started at 1.*

Fact 2 *(Revuz and Yor, [31], p 430) Let* $\{R(t)\}_{t\geq 0}$ *be a Bessel process of dimension* $d > 2$, *starting from* $x > 0$. *Then*

$$\lim_{r\to\infty} \frac{1}{\log r} \int_0^r \frac{\mathrm{d}u}{R^2(u)} = \frac{1}{d-2}, \qquad \mathbb{P}\ a.s. \tag{2.5}$$

The following result was first proved by Dufresne [14] using direct computations. We learned it from Yor [41].

Fact 3 *(Dufresne) Let* $\kappa > 0$. *The law of the almost sure random variable* $S(\infty)$ *is, up to a constant, the inverse of a Gamma distribution. More precisely,*

$$\mathbb{P}\left(S(\infty) \in \mathrm{d}x\right) = \frac{2^{\kappa}}{\Gamma(\kappa)} \mathrm{e}^{-2/x} x^{-(\kappa+1)}\, \mathrm{d}x, \ \text{ for } \qquad x > 0. \tag{2.6}$$

A powerful tool in the study of exponential functionals of drifted Brownian motions, and more generally of Lévy processes, is Lamperti's representation.

Fact 4 *(Lamperti, [27]) Let* $\zeta \in \mathbb{R}$. *There exists* $\{R(t)\}_{t\geq 0}$, *a Bessel process of dimension* $(2+2\zeta)$ *starting from* 2, *such that*

$$\mathrm{e}^{B(t)+\zeta t/2} = \frac{1}{4} R^2 \left(\int_0^t e^{B(y)+\zeta y/2}\, \mathrm{d}y \right), \qquad t \geq 0. \tag{2.7}$$

In particular, taking $\zeta = -\kappa$, $S(\infty)$ appears as the first hitting time of 0 by R. Recall that in this case R is a Bessel process of dimension $2 - 2\kappa$.

Fact 5 *(Getoor and Sharpe, [17]) For all* $z \geq 0$, *and for all* $u \geq 0$ *such that* $2uz < 1$, *we have*

$$\mathbb{E}\left(\exp(uL^z_{\tau(1)})\right) = \exp\left(\frac{u}{1-2uz}\right). \tag{2.8}$$

Moreover,

Fact 6 *(Biane and Yor, [4]) For any $\lambda > 0$, $0 < p < 1$,*

$$\int_0^\infty x^{1/p-2} L^x_{\tau(\lambda)}\, dx \quad \overset{\text{law}}{=} \quad 2p^{2-2/p}\psi(p)\lambda^{1/p}\mathcal{S}_p, \tag{2.9}$$

$$\int_0^1 \frac{L^x_{\tau(1)} - 1}{x}\, dx + \int_1^\infty \frac{L^x_{\tau(1)}}{x} dx \quad \overset{\text{law}}{=} \quad 2\bar{\gamma} + \log\frac{\pi}{4} + \frac{\pi}{2}\mathcal{C}_1, \tag{2.10}$$

where

$$\psi(p) = \left(\frac{\pi p}{4\Gamma^2(p)\sin(\pi p/2)}\right)^{1/p},$$

with $\bar{\gamma}$ denoting Euler's constant, Γ the usual gamma function, $\mathcal{S}_p$ a completely asymmetric stable variable of index p and $\mathcal{C}_1$ a completely asymmetric Cauchy variable of index 1. The laws of $\mathcal{S}_p$ and $\mathcal{C}_1$ are characterized by

$$\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{it\mathcal{S}_p}\right) &= \exp\left(-|t|^p\left(1 - i\,\mathrm{sgn}(t)\tan\frac{\pi p}{2}\right)\right), \\
\mathbb{E}\left(\mathrm{e}^{it\mathcal{C}_1}\right) &= \exp\left(-|t| - it\frac{2}{\pi}\log|t|\right).
\end{aligned}$$

For a reference on stable laws, see e.g. [5], p 347.

Before stating the final result, let us recall the definition of a Jacobi process. See for instance [21].

Definition. *A Jacobi process $\{Y(t),\, t \ge 0, \mathbb{P}\}$ of dimensions (d_1, d_2) starting from $y \in (0,1)$ is the solution of the stochastic differential equation*

$$\mathrm{d}Y(t) = 2\sqrt{Y(t)(1-Y(t))}\ \mathrm{d}\mathbb{B}(t) + (d_1 - (d_1+d_2)Y(t))\,\mathrm{d}t, \tag{2.11}$$

with $0 \le Y(t) \le 1$, $Y(0) = y$ and with $\mathbb{B}$ a $\mathbb{P}$-Brownian motion.

The following result is due to Warren and Yor [38]; it relates Bessel and Jacobi processes.

Fact 7 *(Warren and Yor) Let $\{R_1(t)\}_{t\ge 0}$ and $\{R_2(t)\}_{t\ge 0}$ be two independent Bessel processes of dimensions d_1 and d_2 respectively, with $d_1 + d_2 \ge 2$, $R_1(0) = r_1 \ge 0$ and $R_2(0) = r_2 > 0$. There exists a Jacobi process $\{Y(t)\}_{t\ge 0}$ of dimensions (d_1, d_2), starting from $r_1^2/(r_1^2 + r_2^2)$, independent of $\{R_1^2(t) + R_1^2(t)\}_{t\ge 0}$, and such that for all $t \ge 0$,*

$$\frac{R_1^2(t)}{R_1^2(t) + R_2^2(t)} = Y\left(\int_0^t \frac{\mathrm{d}s}{R_1^2(s) + R_2^2(s)}\right). \tag{2.12}$$

Let us now state and prove three lemmas which will be of constant use in what follows.

3 Three lemmas

Lemma 3.1 *Let $\{Z(t)\}_{t\geq 0}$ denote a Bessel process of dimension 0 started at 1. For all $v, \delta > 0$ and $u \geq 1$, we have*

$$
\begin{aligned}
&(3.1) \qquad & \mathbb{P}\left(\sup_{t\geq 0} Z(t) > u\right) &= \frac{1}{u}, \\
&(3.2) \qquad & \mathbb{P}\left(\sup_{0\leq t\leq v} |Z(t)-1| > \delta\right) &\leq 4\frac{\sqrt{(1+\delta)v}}{\delta} \exp\left(-\frac{\delta^2}{8(1+\delta)v}\right).
\end{aligned}
$$

Set

$$\Sigma(r) \stackrel{def}{=} \int_0^r e^{-W(y)}\,\mathrm{d}y.$$

Intuitively speaking, $\log \Sigma(r)$ is of order $\kappa r/2$. The following lemma gives a rigorous form to this intuition.

Lemma 3.2 *For any $\delta > 0$, there exist two constants c_1 and c_2 depending on both δ and κ, such that, for r big enough,*

$$\mathbb{P}\left(\left|\log \Sigma(r) - \frac{\kappa}{2}r\right| > \delta r\right) \leq c_1 e^{-c_2 r}. \tag{3.3}$$

Furthermore, let $\{R(t)\}_{t\geq 0}$ denote a Bessel process of dimension $d > 2$, starting at 2. For any $\delta > 0$, there exist two constants c_3 and c_4 depending on both δ and d such that, for all r big enough,

$$\mathbb{P}\left(\left|\frac{1}{\log r}\int_0^r \frac{\mathrm{d}s}{R^2(s)} - \frac{1}{d-2}\right| > \delta\right) \leq \frac{c_3}{r^{c_4}}. \tag{3.4}$$

The last result complements (2.5). Still dealing with Bessel processes, the following lemma will be used in Section 7.

Lemma 3.3 *Let $\{R(t)\}_{t\geq 0}$ denote a Bessel process of dimension $d > 2$, starting at $x > 0$. Let $a \geq 0$ and $b > 2a + 2$. For any $p < (d-2)/(b-2)$, we have*

$$\mathbb{E}\left(\int_0^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s\right)^p < \infty.$$

Let us prove the aforestated lemmas; we begin with the

Proof of Lemma 3.1. Recall from (2.4) with $d = 0$ that the process Z solves

$$\mathrm{d}Z(t) = 2\sqrt{Z(t)}\,\mathrm{d}\gamma(t), \tag{3.5}$$

where γ is a standard Brownian motion. The absence of drift in the previous stochastic differential equation makes the function S_Z defined by $S_Z(x) = x$ for all $x \geq 0$ a scale function of Z (one of

many!). Accordingly, the left-hand side of (3.1) is the probability that, starting from 1, Z hits u before hitting 0. This equals $(S_Z(1)-S_Z(0))/(S_Z(u)-S_Z(0))=1/u$, proving (3.1).

As for (3.2), Z is a martingale whose increasing process is $\mathrm{d}<Z,Z>_t=4Z(t)\,\mathrm{d}t$. Thus, by means of the Dubins-Schwarz theorem (cf. [31], p 182), there exists a Brownian motion, say γ^*, starting from 0, such that for all $t\geq 0$,

$$Z(t)-1=\gamma^*\left(4\int_0^t Z(s)\,\mathrm{d}s\right).$$

Setting

$$\begin{aligned}\alpha_\delta &\stackrel{def}{=} \inf\{s>0:\ |Z(s)-1|>\delta\},\\ \sigma_\delta^* &\stackrel{def}{=} \inf\{s>0:\ |\gamma^*(s)|>\delta\},\end{aligned}$$

we get that

$$\sigma_\delta^*=4\int_0^{\alpha_\delta} Z(s)\,\mathrm{d}s\leq 4(1+\delta)\,\alpha_\delta.$$

Thus,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq v}|Z(t)-1|>\delta\right) &= \mathbb{P}\left(\alpha_\delta<v\right)\leq\mathbb{P}\left(\sigma_\delta^*<4(1+\delta)v\right),\\ &= \mathbb{P}\left(\sup_{0\leq s\leq 4(1+\delta)v}|\gamma^*(s)|>\delta\right),\\ &\leq 2\mathbb{P}\left(\sup_{0\leq s\leq 4(1+\delta)v}\gamma^*(s)>\delta\right),\\ &\leq 4\frac{\sqrt{(1+\delta)v}}{\delta}\exp\left(-\frac{\delta^2}{8(1+\delta)v}\right),\end{aligned}$$

as desired. We now move to the

Proof of Lemma 3.2: We start with (3.3). From the definition of $\Sigma(r)$, it is easily seen that for all $r>0$,

$$-\sup_{0\leq s\leq r}B(s)+\log\frac{2}{\kappa}(1-e^{-\kappa r/2})\leq\log\left(\Sigma(r)e^{-\kappa r/2}\right)\leq-\inf_{0\leq s\leq r}B(s)+\log\frac{2}{\kappa}(1-e^{-\kappa r/2}),$$

and since $\kappa>1$, we have

$$-\sup_{0\leq s\leq r}B(s)-\log\kappa\leq\log(\Sigma(r)e^{-\kappa r/2})\leq-\inf_{0\leq s\leq r}B(s)+1$$

for r large enough. Therefore, for such r,

$$\begin{aligned}\mathbb{P}(|\log\Sigma(r)-\kappa r/2|>\delta r) &\leq \mathbb{P}(\sup_{0\leq s\leq r}(-B(s))>\delta r/2)+\mathbb{P}(\sup_{0\leq s\leq r}B(s)>\delta r/2),\\ &= 4\mathbb{P}(B(1)>\delta\sqrt{r}/2)\leq 4\exp(-\frac{\delta^2}{8}r).\end{aligned}$$

We have used the reflection principle together with a Brownian scaling in deriving the equality above. This finishes the proof of (3.3).

The next task is to derive (3.4) from (3.3). Since R is a Bessel process of dimension d starting at 2, according to Lamperti's representation (see 2.7), R can be realized as

$$R(t) = 2\exp\left(-\frac{1}{2}\, W_d(\Sigma_d^{-1}(t))\right), \qquad t \geq 0,$$

where

$$W_d(t) = \gamma(t) - \frac{d-2}{4}\, t, \qquad t \geq 0,$$

with $\{\gamma(t)\}_{t\geq 0}$ a standard Brownian motion, and $\Sigma_d(t) \stackrel{\text{def}}{=} \int_0^t \mathrm{e}^{-W_d(s)}\, \mathrm{d}s$. Accordingly,

$$\int_0^r \frac{\mathrm{d}x}{R^2(x)} = \frac{1}{4}\Sigma_d^{-1}(r).$$

Using the above identity, we get that for all $0 < \delta < 1/(d-2)$, $(d > 2)$, the left-hand side of (3.4) equals

$$\mathbb{P}\left(\frac{\log \Sigma_d(s)}{s} - \frac{d-2}{4} < -\frac{\delta(d-2)^2}{8}\right) + \mathbb{P}\left(\frac{\log \Sigma_d(t)}{t} - \frac{d-2}{4} > \frac{\delta(d-2)^2}{4}\right),$$

where $s = 4(\delta + 1/(d-2))\log r$ and $t = 4(-\delta + 1/(d-2))\log r$. Making $\kappa = (d-2)/2$ in (3.3), one gets that the probability term in (3.4) is less than or equal to c_3 times r^{-c_4}, where c_3 and c_4 are constants depending on d and δ. This finishes the proof of (3.4) and thus that of Lemma 3.2.

Proof of Lemma 3.3: By scaling, we can assume without loss of generality that $x = 2$. Write for any $a \geq 0$,

(3.6) $$Y_a \stackrel{\text{def}}{=} \int_0^\infty \frac{s^a}{R^b(s)}\, \mathrm{d}s.$$

We first study the variable Y_0. By exactly the same means as in the proof of (3.4), namely Lamperti's representation for R, one has

$$Y_0 = \int_0^\infty \frac{\mathrm{d}s}{R^b(s)} = 2^{-b}\int_0^\infty \mathrm{d}\Sigma_d(u)\mathrm{e}^{bW_d(u)/2} = 2^{-b}\int_0^\infty \mathrm{e}^{(b-2)W_d(u)/2}\, \mathrm{d}u.$$

By scaling, this implies that

(3.7) $$Y_0 \stackrel{\text{law}}{=} \frac{1}{2^{b-2}(b-2)^2}\int_0^\infty \mathrm{e}^{\gamma(t)-(d-2)t/2(b-2)}\, \mathrm{d}t.$$

An application of (2.6) with $\kappa = (d-2)/(b-2)$ confirms that

$$\mathbb{E}_2(Y_0^p) < \infty \iff p < \frac{d-2}{b-2}.$$

Now consider the variable Y_a. For any $t>0$,

$$Y_a = \int_0^t \frac{s^a}{R^b(s)}\,\mathrm{d}s + \int_t^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s \le t^a Y_0 + \int_t^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s.$$

For each $p\ge 0$, there exists $d_1(p)$ such that

$$(x+y)^p \le d_1(p)\,(x^p+y^p), \qquad x\ge 0,\ y\ge 0. \tag{3.8}$$

Therefore

$$Y_a^p \le d_1(p)\,t^{ap} Y_0^p + d_1(p)\left(\int_t^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s\right)^p.$$

Recall that $\mathbb{P}_x$ and $\mathbb{E}_x$ denote probability and expectation w.r.t. $\mathbb{P}_x$ when the process starts at x. Taking expectations with respect to $\mathbb{P}_2$ on both sides and using the Markov property, we obtain

$$\mathbb{E}_2(Y_a^p) \le d_1(p)\,t^{ap}\,\mathbb{E}_2(Y_0^p) + d_1(p)\,\mathbb{E}_2\left(\mathbb{E}_{R(t)}\left(\int_0^\infty \frac{(s+t)^a}{R^b(s)}\,\mathrm{d}s\right)^p\right). \tag{3.9}$$

According to (3.8),

$$\begin{aligned}\left(\int_0^\infty \frac{(s+t)^a}{R^b(s)}\,\mathrm{d}s\right)^p &\le d_1^p(a)\left(\int_0^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s + t^a\int_0^\infty \frac{\mathrm{d}s}{R^b(s)}\right)^p \\ &\le d_2\left(\left(\int_0^\infty \frac{s^a}{R^b(s)}\,\mathrm{d}s\right)^p + t^{ap}\left(\int_0^\infty \frac{\mathrm{d}s}{R^b(s)}\right)^p\right),\end{aligned}$$

where $d_2 = d_2(a,p) \stackrel{\text{def}}{=} d_1^p(a)d_1(p)$. Applying the scaling property yields that for any $y>0$,

$$\mathbb{E}_y\left(\int_0^\infty \frac{(s+t)^a}{R^b(s)}\,\mathrm{d}s\right)^p \le d_2\left((y/2)^{(2a-b+2)p}\mathbb{E}_2(Y_a^p) + (2/y)^{(b-2)p}t^{ap}\,\mathbb{E}_2(Y_0^p)\right).$$

Plugging this into (3.9) gives that for any $t>0$,

$$\begin{aligned}\mathbb{E}_2(Y_a^p) \;\le\; & d_1(p)\,t^{ap}\,\mathbb{E}_2(Y_0^p) + d_3\,\mathbb{E}_2(R^{-(b-2a-2)p}(t))\,\mathbb{E}_2(Y_a^p) \\ & + d_4\,t^{ap}\,\mathbb{E}_2(R^{-(b-2)p}(t))\,\mathbb{E}_2(Y_0^p),\end{aligned}$$

with $d_3 = d_3(a,b,p) \stackrel{\text{def}}{=} 2^{-(2a-b+2)p}d_2$ and $d_4 = d_4(a,b,p) \stackrel{\text{def}}{=} 2^{(b-2)p}d_2$. For any $0<u<d$,

$$\mathbb{E}_2(R^{-u}(t)) \le \frac{d_5(u,d)}{t^{u/2}}, \qquad t\ge 1,$$

for some $d_5(u,d)$ (this can be easily checked for example using the exact semi-group of R). Therefore, if $0<p<(d-2)/(b-2)$ which guarantees $0<(b-2a-2)p<d$, then we can choose t sufficiently large that $\mathbb{E}_2(R^{-(b-2a-2)p}(t)) \le 1/(2d_3)$, which yields

$$\mathbb{E}_2(Y_a^p) \le 2d_1(p)\,t^{ap}\,\mathbb{E}_2(Y_0^p) + 2d_4\,t^{ap}\,\mathbb{E}_2(R^{-(b-2)p}(t))\,\mathbb{E}_2(Y_0^p).$$

In particular, this shows that $\mathbb{E}_2(Y_a^p)<\infty$ for all $p<(d-2)/(b-2)$.

On the road to the proof of Theorem 1.1, our first step is to prove Theorem 1.2 for the first hitting time process H. This will be justified in Section 7.2. The following section provides our second step, as finding tail estimates for H amounts to finding tail estimates for two random variables I_1 and Υ, to be defined below.

4 From hitting times to Bessel and Jacobi processes

Recall the definitions of $\{H(r)\}_{r>0}$ and $\{\sigma_{\mathcal{B}}(r)\}_{r>0}$ from (1.5) and (2.2), where $\mathcal{B}$ is a Brownian motion independent of the environment W, see (1.2). In the sequel, we shall drop $\mathcal{B}$ in both $\sigma_{\mathcal{B}}(r)$ and $L_t^x(\mathcal{B})$ for brevity. By (1.2) and the occupation density formula we have, for any $r > 0$,

$$\begin{aligned} H(r) = T(\sigma(S(r))) &= \int_0^{\sigma(S(r))} \mathrm{e}^{-2W(S^{-1}(\mathcal{B}(u)))}\,\mathrm{d}u, \\ &= \int_{-\infty}^{S(r)} e^{-2W(S^{-1}(y))} L^y_{\sigma(S(r))}\,\mathrm{d}y, \\ &= \left(\int_{-\infty}^{0} + \int_0^r\right) e^{-W(x)} L^{S(x)}_{\sigma(S(r))}\,\mathrm{d}x, \\ &\stackrel{def}{=} I_1(r) + I_2(r), \end{aligned}$$

where we have performed the change of variables $x = S^{-1}(y)$ in deriving the third equality.

A scaling argument tells us that, at fixed environment W, the processes $\{L^{S(y)}_{\sigma(S(r))}\}_{y\le r}$ and $\{S(r)\, L^{S(y)/S(r)}_{\sigma(1)}\}_{y\le r}$ have the same law.

Further, according to the first Ray–Knight theorem (see Fact 1, Section 2), $\{L^{-z}_{\sigma(1)}\}_{z\ge 0}$ is a squared Bessel process of dimension 0, with initial exponential distribution ξ of mean 2 and $\{L^{1-t}_{\sigma(1)}\}_{0\le t\le 1}$ is a two-dimensional squared Bessel process starting from 0 and independent of W, say R_1^2.

As result, $H(r)$ can be rewritten as:

$$H(r) \stackrel{law}{=} S(r)\,\xi \int_{-\infty}^{0} \mathrm{e}^{-W(y)}\, Z\left(\frac{|S(y)|}{S(r)\,\xi}\right) \mathrm{d}y + S(r) \int_0^r R_1^2\left(1 - \frac{S(y)}{S(r)}\right) \mathrm{d}y, \tag{4.1}$$

where $\{Z(t)\}_{t\ge 0}$ is a squared Bessel process of dimension 0, starting from 1, and where $S(r)$, $\{W(y)\}_{y\le 0}$, ξ and $\{Z(t)\}_{t\ge 0}$ are independent.

Note that the above identity in law is *quenched*, hence also annealed.

A glance at the definition of $I_1(r)$ tells us that $r \mapsto I_1(r)$ increases so that, $\mathbb{P}$-almost surely,

$$\sup_{r\ge 0} I_1(r) = I_1(\infty) < \infty \tag{4.2}$$

Actually, since $\kappa > 1$, both $S(\infty)$ and $\int_{-\infty}^{0} e^{-W(y)}\,\mathrm{d}y$ have finite expectations; it is then easily checked, from (4.1), that $\mathbb{E}(I_1(\infty)) < \infty$.

Now, in order to estimate the tail probabilities of $I_2(r)$, a slight transformation of the expression given in (4.1) is needed.

At fixed W, a scaling argument used twice, followed by the change of variables $z = r - y$, leads to the following series of quenched identities in law:

$$
\begin{aligned}
I_2(r) &\overset{law}{=} S(r)\int_0^r e^{-W(y)}R_1^2\left(1-\frac{S(y)}{S(r)}\right)\,\mathrm{d}y,\\
&\overset{law}{=} \int_0^r e^{-W(y)}R_1^2(S(r)-S(y))\,\mathrm{d}y,\\
&\overset{law}{=} \int_0^r e^{W(r)-W(y)}\,R_1^2\left(\int_y^r e^{-(W(r)-W(x))}\,\mathrm{d}x\right),\\
&= \int_0^r e^{B_z^r-\kappa z/2}R_1^2\left(\int_0^z e^{-(B_x^r-\kappa x/2)}\,\mathrm{d}x\right)\,\mathrm{d}z,
\end{aligned}
$$

where $B_z^r = B(r)-B(r-z)$, for $0\le z\le r$. Since B^r and B have the same law on $[0,r]$, one gets the following *annealed* identity in law:

$$
I_2(r) \overset{law}{=} \int_0^r e^{W(y)}R_1^2\left(\int_0^y e^{-W(x)}\,\mathrm{d}x\right)\,\mathrm{d}y.
$$

Using Lamperti's representation (see (2.7) with $\zeta=\kappa$) gives that

$$
e^{-W(y)} = \frac{1}{4}\,R_2^2\left(\int_0^y e^{-W(x)}\,\mathrm{d}x\right),
$$

where R_2 is a transient Bessel process of dimension $2+2\kappa$, starting from 2. In this light, denoting for simplicity $\int_0^y e^{-W(x)}\,\mathrm{d}x$ by $\Sigma(y)$, then performing the change of variables $u=\Sigma(y)$, we arrive at:

$$
I_2(r) \overset{law}{=} 16\int_0^{\Sigma(r)} \frac{R_1^2(u)}{R_2^4(u)}\,\mathrm{d}u \overset{def}{=} 16 I_3(\Sigma(r)). \tag{4.3}
$$

Observe that R_2 depends only on the environment W (of which R_1 is independent).

According to a result by Warren and Yor, see (2.12), there exists a Jacobi process of dimensions $(2, 2+2\kappa)$, say Y, starting from 0, such that

$$
I_3(r) \overset{law}{=} \int_0^{\Lambda(r)} \frac{Y(s)}{(1-Y(s))^2}\,\mathrm{d}s \overset{\mathrm{def}}{=} \Upsilon(\Lambda(r)), \tag{4.4}
$$

where

$$
\Upsilon(r) \overset{\mathrm{def}}{=} \int_0^r \frac{Y(s)}{(1-Y(s))^2}\,\mathrm{d}s, \tag{4.5}
$$

$$
\Lambda(r) \overset{\mathrm{def}}{=} \int_0^r \frac{\mathrm{d}u}{R_1^2(u)+R_2^2(u)}. \tag{4.6}
$$

Note that since $R^2 \overset{def}{=} R_1^2+R_2^2$, the process R is a squared Bessel process of dimension $d \overset{def}{=} 4+2\kappa > 2$, starting from 2. We know from (2.5) that $\Lambda(r)/\log r$ approaches, $\mathbb{P}$-almost surely, $1/(d-2) = 1/(2+2\kappa)$. Moreover, Lemma 3.2 makes us expect that tail estimates for I_2 will follow from those for Υ. And they do:

Theorem 4.1 *For all $u > (1+\kappa)/(\kappa(\kappa-1))$,*

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}\left(\Upsilon(r) > ur\right) \le 1-\kappa, \tag{4.7}$$

and for all $u > 0$,

$$\liminf_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}\left(\Upsilon(r) > ur\right) \ge 1-\kappa. \tag{4.8}$$

Moreover, for any $w > 0$,

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}(I_1(\infty) > wr) \le 1-\kappa. \tag{4.9}$$

Proving that Theorem 1.2 follows from Theorem 4.1: Assuming that both (4.7) and (4.9) hold true, we first prove (1.6), the upper bound for H.

To this end, let $u > v_\kappa^{-1} = 4/(\kappa-1)$ be given. Then there exists some δ_0 such that $u > (1+\delta_0)v_\kappa^{-1}$. Next pick ϵ with $0 < \epsilon < \delta_0/(1+\delta_0)$ so that $(1-\epsilon)(1+\delta_0) > 1$, then choose $\delta > 0$ so small that $(1-\epsilon)(1+\delta_0) > 1+2\delta$.

Now, since $H(r) = I_1(r) + I_2(r)$, using successively the triangle inequality, (4.2), the definition of I_3, (4.3), the fact that $r \mapsto I_3(r)$ is increasing, then finally Lemma 3.2, (3.3), it follows that

$$\begin{aligned}\mathbb{P}(H(r) > ur) &\le \mathbb{P}(I_1(r) > \epsilon ur) + \mathbb{P}(I_2(r) > (1-\epsilon)ur),\\ &\le \mathbb{P}(I_1(\infty) > \epsilon ur) + \mathbb{P}(I_2(r) > (1-\epsilon)ur),\end{aligned}$$

with

$$\begin{aligned}\mathbb{P}(I_2(r) > (1-\epsilon)ur) &= \mathbb{P}\left(16\, I_3(\Sigma(r)) > (1-\epsilon)ur\right),\\ &\le \mathbb{P}\left(I_3(e^{\kappa(1+2\delta)r/2}) > \frac{(1-\epsilon)}{16}ur\right) + c_1 e^{-c_2 r},\\ &= \mathbb{P}\left(I_3(s) > v\log s\right) + c_1 e^{-c_2 r},\end{aligned}$$

where $s = s(r) = e^{\kappa(1+2\delta)r/2}$ and

$$v = \frac{1-\epsilon}{1+2\delta} \times \frac{u}{8\kappa} > \frac{1}{2\kappa(\kappa-1)} \times \frac{(1-\epsilon)(1+\delta_0)}{1+2\delta} > \frac{1}{2\kappa(\kappa-1)},$$

thanks to the choices of δ_0, ϵ and δ.

Now, since $v > (2\kappa(\kappa-1))^{-1}$, there exists $0 < \epsilon_0 < 1$ such that $v > (1+\epsilon_0)/2\kappa(\kappa-1)$. Since $4+2\kappa > 2$, (3.4) tells us about the rate at which $\Lambda(s)/\log s$ approaches $\mathbb{P}$-almost surely $1/(2+2\kappa)$ as s tends to infinity. Knowing (4.4), this in conjunction with the fact that Υ is increasing yields

$$\begin{aligned}\mathbb{P}(I_3(s) > v\log s) &\le \mathbb{P}(\Upsilon(t) > v\log s) + \mathbb{P}\left(\left|\frac{\Lambda(s)}{\log s} - \frac{1}{2+2\kappa}\right| > \frac{\epsilon_0}{2+2\kappa}\right),\\ &\le \mathbb{P}(\Upsilon(t) > wt) + \frac{c_3}{s^{c_4}},\end{aligned}$$

for all $n \geq 1$, and for some constants c_3 and c_4 depending on ϵ_0 and κ, where

$$t = t(s) = \frac{1+\epsilon_0}{2+2\kappa}\log s, \quad w = \frac{2+2\kappa}{1+\epsilon_0}v > \frac{1+\kappa}{\kappa(\kappa-1)}.$$

Consequently, putting all the pieces together, and keeping in mind that t is actually r times a constant depending on δ, ϵ_0 and κ, one gets

$$\mathbb{P}(H(r) > ur) \leq \mathbb{P}(I_1(\infty) > \epsilon ur) + \mathbb{P}(\Upsilon(t) > wt) + c_3 e^{-c_4 r/2}, \tag{4.10}$$

for $\kappa > 1 > \delta > 0$. Taking the logarithm of both sides of (4.10), using the elementary fact that $\log(a+b+c) \leq \log 3 + \sup(\log a, \log b, \log c)$ for $a, b, c > 0$, dividing by $\log r$ ($\log r \sim \log t$, as $r \to \infty$), taking the limsup, and making use of (4.7) and (4.9) completes the proof of the upper bound for H (1.6).

As for the lower bound for H (1.7), this follows from the lower bound for Υ (4.8), by Lemma 3.2. The reasoning is the same as before (but a bit simpler actually since $I_1(\infty)$ does not enter the picture): we write $\mathbb{P}(H(r) > ur) \geq \mathbb{P}(I_2(r) > ur) = \mathbb{P}(16 I_3(\Sigma(r)) > ur)$, and use the same arguments as before.

This indicates how (1.7) follows from (4.8).

We have seen that estimating tail probabilities for $H(r)$ reduces to proving Theorem 4.1. This amounts to studying the tail asymptotics for $I_1(\infty)$ and Υ. We postpone the study of $I_1(\infty)$ to Section 7 and move on to the proofs of (4.7) and (4.8) of Theorem 4.1. In the next section, we state and prove a key result which will enable us to prove these results.

5 A key estimate

From the stochastic differential equation (2.11), we have

$$\mathrm{d}S_Y(x) = \frac{\mathrm{d}x}{x(1-x)^{\kappa+1}}, \quad 0 < x < 1,$$

and

$$m_Y(dx) = \frac{1}{4}(1-x)^{\kappa}\,\mathrm{d}x, \quad 0 < x < 1,$$

where S_Y is a scale function of the diffusion Y and m_Y is its speed measure.

Recall that $Y_0 = 0$ and, given the definitions of R_1 and R_2, that $0 < Y(t) < 1$ for all $t > 0$, by (2.12).

It is easily checked that Y is recurrent, and that 0 and 1 are actually entrance boundaries (see [21], page 235); they cannot be reached from]0,1[. Since in our case Y starts at 0, it *rapidly* moves to]0,1[never to return to 0. Tail estimates for the first hitting time of level 1/2 by the diffusion Y, started at 0, shed some light on just how fast Y moves from 0 to 1/2; see Lemma (6.1).

In order to establish (4.7) and (4.8) of Theorem 4.1, we first assume that Y starts in]0,1[, choosing without loss of generality that it starts at 1/2, and get the desired estimates with $\mathbb{P}_{1/2}$

replacing $\mathbb{P}$. Next, as is proved in the next section, Lemma 6.1 in conjunction with the strong Markov property enables us to establish the result for the case where Y starts at 0; this then yields (4.7) and (4.8).

Since $Y(0) = 1/2$, a scale function of Y is

$$S_Y(x) = \int_{1/2}^{y} \frac{dx}{x(1-x)^{\kappa+1}}, \quad y \in (0,1), \tag{5.1}$$

so that

$$dS_Y(Y(t)) = 2Y^{-1/2}(t)(1-Y(t))^{-\kappa-1/2} d\mathbb{B}(t). \tag{5.2}$$

Thus, Y can be constructed from a Brownian motion via a scale transformation and time change. Namely, there exists a driftless Brownian motion β such that

$$S_Y(Y(t)) = \beta(U(t)), \quad t \geq 0, \tag{5.3}$$

where the time change U is given by

$$U(t) \stackrel{\text{def}}{=} 4 \int_0^t \frac{ds}{Y(s)(1-Y(s))^{2\kappa+1}}. \tag{5.4}$$

From [19] we get that, for a certain Brownian motion β_r defined below, see (5.7), $U(r)/r^2$ is roughly $\tau_{\beta_r}(4(\kappa+1))$. The next proposition provides a key estimate which measures the error introduced by this replacement. Before stating this, we need some definitions. Let α, δ, μ, $\nu > 0$ such that $\delta < 1$ and $\mu + \nu < 1$, and set

$$\lambda_{\pm} \stackrel{\text{def}}{=} \lambda(1 \pm \delta), \qquad \text{where} \quad \lambda \stackrel{\text{def}}{=} 4(\kappa+1), \tag{5.5}$$

$$0 < \theta < 1, \quad \mu = e^{-r^{\theta}} \;\; \textit{and} \;\; \nu = r^{-\theta/\kappa}, \tag{5.6}$$

$$\beta_r(s) \stackrel{\text{def}}{=} \frac{1}{r}\beta(r^2 s), \qquad s \geq 0, \tag{5.7}$$

$$\epsilon(r,s) \stackrel{\text{def}}{=} \frac{1}{4}\int_0^1 (1-x)^{\kappa} \left(L_s^{S_Y(x)/r}(\beta_r) - L_s^0(\beta_r) \right) \mathrm{d}x. \tag{5.8}$$

and

$$\Xi_r = \Xi(r,\kappa,\delta) \stackrel{\text{def}}{=} \left\{ \epsilon(r, \tau_{\beta_r}(\lambda_+)) \geq -\delta \; ; \; \epsilon(r, \tau_{\beta_r}(\lambda_-)) \leq \delta \right\}. \tag{5.9}$$

So as not to overburden the reader with notation, we shall drop β_r in both $\tau_{\beta_r}(.)$ and $L^{\cdot}_{\tau_{\beta_r}(.)}(\beta_r)$ throughout.

Proposition 5.1 *Let $0 < \delta < 1$ be given. On the event Ξ_r, we have, for all $r > 0$,*

$$\tau(\lambda_-) \leq \frac{U(r)}{r^2} \leq \tau(\lambda_+), \tag{5.10}$$

and

$$\lim_{r\to\infty} \frac{\log \mathbb{P}_{1/2}\left(\Xi_r^c\right)}{\log r} = -\infty, \tag{5.11}$$

where Ξ_r^c stands for the complement of Ξ_r.

Proof of Proposition 5.1 Let U^{-1} denote the inverse of U. Thanks to (5.3), the density occupation formula (2.1), and (5.4), we arrive (exactly as in [19] with $d_1 = 2$ and $d_2 = 2 + 2\kappa$) at the following:

$$\begin{aligned} U^{-1}(t) &= \frac{1}{4}\int_0^t \left(S_Y^{-1}(\beta(s))\right)\,\left(1 - S_Y^{-1}(\beta(s))\right)^{1+2\kappa}\,\mathrm{d}s \\ &= \frac{1}{4}\int_0^1 (1-x)^\kappa L_t^{S_Y(x)}(\beta)\,\mathrm{d}x. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{r}U^{-1}(r^2 s) &= \frac{1}{4}\int_0^1 (1-x)^\kappa\, L_s^{S_Y(x)/r}(\beta_r)\,\mathrm{d}x, \\ &= \frac{1}{\lambda}L_s^0(\beta_r) + \epsilon(r,s). \end{aligned}$$

Since U is increasing, (5.9) together with straightforward computations delivers (5.10).

We now turn to (5.11). Recalling from (5.5) and (5.8) the definitions of λ_+ and $\epsilon(.,.)$ respectively, we have that for all μ and ν positive such that $\mu + \nu < 1$,

$$\begin{aligned} \mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_+)) < -\delta\right) &\leq \mathbb{P}\left(\int_\mu^{1-\nu} (1-x)^\kappa L_{\tau(\lambda_+)}^{S_Y(x)/r}\,\mathrm{d}x < 4\right), \\ &\leq \mathbb{P}\left(\inf_{\frac{S_Y(\mu)}{r\lambda_+}\leq x\leq \frac{S_Y(1-\nu)}{r\lambda_+}} L_{\tau(1)}^x < 1-\delta_1\right), \end{aligned}$$

where

$$\delta_1 = \delta_1(\delta,\mu,\nu,\kappa) = 1 - \frac{1}{(1+\delta)\left((1-\mu)^{\kappa+1} - \nu^{\kappa+1}\right)}.$$

In deriving the last inequality, we have used a scaling argument together with the monotonicity of S_Y. A little Brownian excursion theory now tells us that $\{L_{\tau(1)}^x\}_{x\geq 0}$ and $\{L_{\tau(1)}^{-x}\}_{x\geq 0}$ are two

independent squared Bessel processes of dimension 0, started at 1, (see [31]). Therefore the last probability above is

$$\begin{aligned}
&\le\ 2\,\mathbb{P}\left(\inf_{0\le x\le \frac{S_Y(1-\nu)\vee|S_Y(\mu)|}{r\lambda_+}} L^x_{\tau(1)} < 1-\delta_1\right),\\
&\le\ 2\,\mathbb{P}\left(\sup_{0\le x\le \frac{S_Y(1-\nu)\vee|S_Y(\mu)|}{r\lambda_+}} |L^x_{\tau(1)} - 1| > \delta_1\right).
\end{aligned}$$

Recalling the definition of S_Y, (5.1), straightforward computations tell us that

$$\begin{aligned}
2^{\kappa+1}\log 2x &\le S_Y(x) \le \frac{1}{(1-x)^{\kappa+1}}\log 2x,\\
\frac{1}{2\kappa x^{\kappa}} &\le S_Y(1-x) \le \frac{2}{x^{\kappa}},
\end{aligned}\tag{5.12}$$

as x approaches 0.

Thus, with the choices of μ and ν, see (5.6), we get

$$\frac{1}{r\lambda_+}(S_Y(1-\nu)\vee|S_Y(\mu)|) \le 2^{\kappa-2}r^{\theta-1},$$

for r large enough. Accordingly, with the help of (3.2),

$$\begin{aligned}
2\mathbb{P}\left(\sup_{0\le x\le \frac{S_Y(1-\nu)\vee|S(\mu)|}{r\lambda_+}} |L^x_{\tau(1)} - 1| > \delta_1\right) &\le 2\mathbb{P}\left(\sup_{0\le x\le 2^{\kappa-2}r^{\theta-1}} |L^x_{\tau(1)} - 1| > \delta_1\right),\\
&\le f_1 \mathrm{e}^{-f_2 r^{1-\theta}},
\end{aligned}$$

where $f_1 = 5\,2^{\kappa/2}/\delta_1$ and $f_2 = \delta_1^2\,2^{-\kappa-2}$.

As a result,

$$\lim_{r\to\infty}\frac{1}{\log r}\log \mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_+)) < -\delta\right) = -\infty.\tag{5.13}$$

We now turn to $\mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_-)) > \delta\right)$. It is plain to see that

$$\mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_-)) > \delta\right) \le \mathbb{P}_{1/2}(J_1(r,\mu) > 2\delta) + \mathbb{P}_{1/2}(J_2(r,\mu,\nu) > \delta) + \mathbb{P}_{1/2}(J_3(r,\nu) > \delta).\tag{5.14}$$

where

$$\begin{aligned}
J_1(r,\mu) &\stackrel{def}{=} \int_0^{\mu}(1-x)^{\kappa}\left(L^{S_Y(x)/r}_{\tau(\lambda_-)} - \lambda_-\right)\mathrm{d}x,\\
J_2(r,\mu,\nu) &\stackrel{def}{=} \int_{\mu}^{1-\nu}(1-x)^{\kappa}\left(L^{S_Y(x)/r}_{\tau(\lambda_-)} - \lambda_-\right)\mathrm{d}x,\\
J_3(r,\nu) &\stackrel{def}{=} \int_{1-\nu}^{1}(1-x)^{\kappa}\left(L^{S_Y(x)/r}_{\tau(\lambda_-)} - \lambda_-\right)\mathrm{d}x.
\end{aligned}$$

We start with $\mathbb{P}_{1/2}(J_1(r,\mu) > 2\delta)$. Since $\sup_{x\le 0} L^x_{\tau(1)}$ and $\sup_{x\ge 0} L^x_{\tau(1)}$ have the same law, (3.1) together with a scaling leads to

$$\begin{aligned}
\mathbb{P}_{1/2}(J_1(r,\mu) > 2\delta) &\le \mathbb{P}\left(\sup_{x\le 0} L^x_{\tau(1)} > \frac{\delta}{\lambda_-\mu}\right), \\
&= \mathbb{P}\left(\sup_{x\ge 0} L^x_{\tau(1)} > \frac{\delta}{\lambda_-\mu}\right) = \frac{\lambda_-}{\delta}\mu = \frac{\lambda_-}{\delta}\, e^{-r^\theta}. \qquad (5.15)
\end{aligned}$$

Next, we may write

$$\mathbb{P}_{1/2}(J_2(r,\mu,\nu) > \delta) \le 2\,\mathbb{P}\left(\sup_{0\le x\le (S_Y(1-\nu)\vee|S_Y(\mu)|)/\lambda_- r} \left|L^x_{\tau(1)} - 1\right| > \delta_2\right),$$

where

$$\delta_2 = \delta_2(\delta,\mu,\nu,\kappa) = \frac{\delta}{4(1-\delta)\left((1-\mu)^{\kappa+1} - \nu^{\kappa+1}\right)}.$$

Thus, for r large, with the same choice of μ and ν as before, one gets

$$\mathbb{P}_{1/2}(J_2(r,\mu,\nu) > \delta) \le f_3 \mathrm{e}^{-f_4\, r^{1-\theta}}. \qquad (5.16)$$

where $f_3 = 2^{2+\kappa/2}\sqrt{1+\delta_2}\delta_2^{-1}$, and $f_4 = \delta_2^2\, 2^{-\kappa-1}(1+\delta_2)^{-1}$.

Lastly, we find an upper bound for $\mathbb{P}_{1/2}(J_3(r,\nu) > \delta)$. Thanks to (5.12), $1 - S_Y^{-1}(y) \le (2/y)^{1/\kappa}$ for y large enough. As a result, performing the change of variables $y = S_Y(x)/r$ together with a change of scale yields

$$\begin{aligned}
\mathbb{P}_{1/2}(J_3(r,\nu) > \delta) &\le \mathbb{P}\left(\int_{1-\nu}^1 (1-x)^\kappa\, L^{S_Y(x)/r}_{\tau(\lambda_-)}\,\mathrm{d}x > \delta\right), \\
&\le \mathbb{P}\left(\int_{fr^{\theta-1}}^\infty \frac{L^y_{\tau(1)}}{y^{2+1/\kappa}}\,\mathrm{d}y > \delta_3 r^{1+1/\kappa}\right), \qquad (5.17)
\end{aligned}$$

where $f = (2\kappa\,\lambda_-)^{-1}$ and $\delta_3 = \frac{\delta}{4}(\lambda_-/2)^{1/\kappa}$.

Now, the definition of a Bessel process of dimension 0, see (3.5), together with the integration by parts

$$\frac{L^y_{\tau(1)}}{y^{2+1/\kappa}}dy = -d\left(L^y_{\tau(1)}\frac{1}{(1+1/\kappa)y^{1+1/\kappa}}\right) + 2\frac{\sqrt{L^y_{\tau(1)}}}{(1+1/\kappa)y^{1+1/\kappa}}\,\mathrm{d}\gamma(y),$$

implies that the last probability above is

$$\le \mathbb{P}\left(L^{fr^{\theta-1}}_{\tau(1)} > \delta_4 r^{\theta(1+1/\kappa)}\right) + \mathbb{P}\left(\int_{fr^{\theta-1}}^\infty \frac{\sqrt{L^y_{\tau(1)}}}{y^{1+1/\kappa}}\,\mathrm{d}\gamma(y) > \delta_5 r^{1+1/\kappa}\right),$$

where $\delta_4 = \frac{\delta_3}{2}(1+\frac{1}{\kappa})f^{1+1/\kappa}$ and $\delta_5 = \delta_3(1+1/\kappa)/4$. Recall that $L^y_{\tau(1)}$ almost surely goes to 0 as y goes to infinity.

An exponential inequality together with (2.8) for $z = fr^{\theta-1}$ and $u = 1/(3z)$ tells us that

$$\mathbb{P}\left(L_{\tau(1)}^{fr^{\theta-1}} > \delta_4 r^{\theta(1+1/\kappa)}\right) \le e^{-\frac{\delta_4}{4f}\, r^{1+\theta/\kappa}},$$

for r large enough. On the other hand, writing the stochastic integral above as a time changed Brownian motion gives, again for r sufficiently large,

$$\mathbb{P}\left(\int_{fr^{\theta-1}}^{\infty} \frac{\sqrt{L_{\tau(1)}^{y}}}{y^{1+1/\kappa}}\, \mathrm{d}\gamma(y) \ge \delta_5 r^{1+1/\kappa}\right) \le e^{-\frac{\delta_5^2}{2}\log^2 r} + \mathbb{P}\left(\int_{fr^{\theta-1}}^{\infty} \frac{L_{\tau(1)}^{y}}{y^{2+2/\kappa}}\, \mathrm{d}y > \frac{r^{2+2/\kappa}}{\log^2 r}\right).$$

The last term is nothing but $\mathbb{P}(\sup_{0\le y\le 1}\gamma(y) > \delta_5\, \log r) = 2\ \mathbb{P}(\mathcal{N} > \delta_5\, \log r)$, thanks to the reflection principle, see [31], for $\mathcal{N}$ a normalized Gaussian variable.

The iteration scheme:

We iterate the procedure above m times, which gives that $\mathbb{P}_{1/2}(J_3(r,\nu) > \delta)$ is

$$\le m e^{-f_m \log^2 r} + m e^{-\frac{\delta_4}{4f}\, r^{1+\theta/\kappa}} + \mathbb{P}\left(\int_{fr^{\theta-1}}^{\infty} \frac{L_{\tau(1)}^{y}}{y^{2+2^m/\kappa}}\, \mathrm{d}y > \frac{r^{2^m+2^m/\kappa}}{(\log r)^{2^{m+1}-2}}\right), \tag{5.18}$$

where f_m is a constant depending only on f and the integer m, or equivalently, on κ, δ and m.

Since $\{L_{\tau(1)}^{y} - 1\}_{y\ge 0}$ is a martingale, see for instance [31], $\mathbb{E}(L_{\tau(1)}^{y}) = 1$ for all $y \ge 0$, and thus, by Chebychev's inequality

$$\mathbb{P}\left(\int_{fr^{\theta-1}}^{\infty} \frac{L_{\tau(1)}^{y}}{y^{2+2^m/\kappa}}\, \mathrm{d}y > \frac{r^{2^m+2^m/\kappa}}{(\log r)^{2^{m+1}-2}}\right) \le f^{-1-2^m/\kappa}\ \frac{(\log r)^{2^{m+1}-2}}{r^{2^m(1+\theta/\kappa)+\theta-1}}. \tag{5.19}$$

Recall that (5.13) takes care of $\mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_+)) < -\delta\right)$. On the other hand, putting (5.15), (5.16) and (5.19) together gives that

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}_{1/2}\left(\epsilon(r,\tau(\lambda_-)) > \delta\right) \le -\left(2^m(1+\theta/\kappa) + \theta - 1\right),$$

for all $0 < \theta < 1$ and any fixed but arbitrary integer m. Letting m go to infinity gives that the limsup above is in fact a limit, which equals $-\infty$. The proof of (5.11) is now complete. We are ready for the

6 Tail estimates for Υ

As announced in the beginning of last section, getting tail estimates for Υ will split into two parts: we first deliver the desired result assuming that $Y(0) = 1/2$, then show how to transfer the result to the case where $Y(0) = 0$. We start with:

6.1 The case $Y(0) = 1/2$

Having in mind (4.5), (5.4), (5.3) and (5.7), one can write

$$\begin{aligned}\Upsilon(r) &= \frac{1}{4}\int_0^r Y^2(s)(1-Y(s))^{2\kappa-1}\,\mathrm{d}U(s),\\ &= \frac{1}{4}\int_0^{U(r)} \left(S_Y^{-1}(\beta(s))\right)^2\left(1-S_Y^{-1}(\beta(s))\right)^{2\kappa-1}\,\mathrm{d}s,\\ &= \frac{1}{4}\int_{S_Y(0)}^{\infty} \left(S_Y^{-1}(s)\right)^2\left(1-S_Y^{-1}(s)\right)^{2\kappa-1} L^s_{U(r)}(\beta)\,\mathrm{d}s,\\ &= \frac{1}{4}\int_0^1 s(1-s)^{\kappa-2} L^{S_Y(s)}_{U(r)}(\beta)\,\mathrm{d}s,\\ &= \frac{r}{4}\int_0^1 s(1-s)^{\kappa-2} L^{S_Y(s)/r}_{U(r)/r^2}(\beta_r)\,\mathrm{d}s.\end{aligned}$$

We have successively used the occupation density formula and a scaling argument in deriving the last two identities. We begin with

The lower bound. Let $u, \delta > 0$ be given. The last identity above coupled with Proposition 5.1, (5.12) and a scaling leads to

$$\begin{aligned}\mathbb{P}_{1/2}(\Upsilon(r) > ur) &\geq \mathbb{P}\left(\int_{1-\nu}^1 s(1-s)^{\kappa-2} L^{S_Y(s)/r}_{\tau(\lambda_-)} ds > 4u\right) - \mathbb{P}_{1/2}(\Xi_r^c),\\ &\geq \mathbb{P}\left(\int_{\frac{2}{\lambda_- r\nu^\kappa}}^{\infty} \frac{L^y_{\tau(1)}}{y^{2-1/\kappa}} dy > u_1 r^{1-1/\kappa}\right) - \mathbb{P}_{1/2}(\Xi_r^c), \qquad (6.1)\end{aligned}$$

where $0 < \nu < 1$ and

$$u_1 = u_1(\delta, \kappa, \nu, u) = \frac{2^{4-1/\kappa}\kappa^{2-1/\kappa}\lambda_-{}^{1/\kappa}}{(1-\nu)^2}\,u.$$

Now, let l be a real number such that $1 < \kappa < l$. The Cauchy-Schwarz inequality gives

$$\left(\int_{\frac{2}{\lambda_- r\nu^\kappa}}^{\infty} \frac{L^y_{\tau(1)}}{y^{2-l/\kappa}}\,\mathrm{d}y\right)^2 \leq \left(\int_{\frac{2}{\lambda_- r\nu^\kappa}}^{\infty} \frac{L^y_{\tau(1)}}{y^{2-1/\kappa}}\,\mathrm{d}y\right)\left(\int_0^{\infty} \frac{L^y_{\tau(1)}}{y^{2-(2l-1)/\kappa}}\,\mathrm{d}y\right).$$

Thus, setting $q \overset{\text{def}}{=} 1 - 1/\kappa$, the first probability in (6.1) is, for all $\eta > 0$,

$$(6.2) \qquad \geq \mathbb{P}\left(\int_{\frac{2}{r\lambda_-\nu^\kappa}}^{\infty} \frac{L^y_{\tau(1)}}{y^{2-l/\kappa}}\,\mathrm{d}y > \sqrt{u_1}\, r^{lq+\frac{2l-1}{2}\eta}\right) - \mathbb{P}\left(\int_0^{\infty} \frac{L^y_{\tau(1)}}{y^{2-(2l-1)/\kappa}}\,\mathrm{d}y > r^{(2l-1)(q+\eta)}\right).$$

By virtue of (2.9), the second probability in (6.2) involves a stable random variable of parameter $0 < \kappa/(2l-1) < 1$ and hence is equivalent to $r^{1-\kappa-\eta\kappa}$. Indeed, from [5], p 347, we have that for $\mathcal{S}_\alpha$ a stable random variable of index $0 < \alpha < 1$, then for x large enough, $\mathbb{P}(\mathcal{S}_\alpha > x)$ is of order $x^{-\alpha}$.

(We say that $u(x)$ is *of order* $v(x)$ as x tends to infinity when $\lim_{x\to\infty} u(x)/v(x)$ equals some finite nonzero constant.)

On the other hand, the first one is, for all $\epsilon > 0$,

$$\geq \mathbb{P}\left(\int_0^\infty \frac{L^y_{\tau(1)}}{y^{2-l/\kappa}}\,\mathrm{d}y > (\sqrt{u_1}+\epsilon) r^{lq+\frac{2l-1}{2}\eta}\right) - \mathbb{P}\left(\int_0^{\frac{2}{\lambda_- r\nu^\kappa}} \frac{L^y_{\tau(1)}}{y^{2-l/\kappa}}\,\mathrm{d}y > \epsilon r^{lq+\frac{2l-1}{2}\eta}\right),$$

$$\geq \mathbb{P}\left(\int_0^\infty \frac{L^y_{\tau(1)}}{y^{2-l/\kappa}}\,\mathrm{d}y > (\sqrt{u_1}+\epsilon) r^{lq+\frac{2l-1}{2}\eta}\right) - \mathbb{P}\left(\sup_{y\geq 0} L^y_{\tau(1)} > \epsilon\, \nu^{l-\kappa}\, r^{l-1+\frac{2l-1}{2}\eta}\right). \tag{6.3}$$

Once again, since the first probability in (6.3) involves a stable variable of parameter $\kappa/l < 1$, it is of order $r^{1-\kappa-\kappa\eta(1-1/2l)}$.

Finally, choosing $\nu = r^{-1+(\eta\kappa)/(l-\kappa)}$ and making use of (3.1) tell us that the second probability in (6.3) is equal to $\epsilon^{-1} r^{1-\kappa-\eta(\kappa+(2l-1)/2)}$. Putting all that together and having in mind (5.11), and the fact that u_1 approaches a constant $u_1(\delta,\kappa,o,u) = 2^{4-1/\kappa}\kappa^{2-1/\kappa}\lambda_-{}^{1/\kappa}$, as r goes to infinity, we see that $\mathbb{P}_{1/2}(\Upsilon(r) > ur)$ is bounded from below by some constant times $r^{1-\kappa-\kappa\eta(1-1/2l)}$, for r big enough, with $\eta, \epsilon > 0$ as small as desired. This completes the proof of (4.8). We now move to

The upper bound. Let $u > u^0 \stackrel{def}{=} \kappa + 1/(\kappa(\kappa-1))$ be given. There exists $\delta_0 = \delta_0(u) > 0$ such that for all $0 < \delta < \delta_0$, $u > u^0_+ \stackrel{def}{=} u^0(1+\delta)$. Hence, for such a δ, by virtue of (5.10),

$$\mathbb{P}_{1/2}(\Upsilon(r) > ur) \leq \mathbb{P}\left(\int_0^1 x(1-x)^{\kappa-2} L^{S_Y(x)/r}_{\tau(\lambda_+)}\,\mathrm{d}x > 4u\right) + \mathbb{P}_{1/2}(\Xi^c_r).$$

Just as in the proof of Proposition 5.1, with the same μ and $\nu > 0$ as before, namely $\mu = e^{-r^\theta}$ and $\nu = r^{-\theta/\kappa}$, for $0 < \theta < 1$, we have

$$\mathbb{P}_{1/2}(\Upsilon(r) > ur) \leq \text{(I)} + \text{(II)} + \text{(III)} + \mathbb{P}_{1/2}(\Xi^c_r),$$

where

$$\text{(I)} \stackrel{def}{=} \mathbb{P}\left(\sup_{x\geq 0} L^x_{\tau(1)} > \frac{4(u-u^0_+)(1-\mu)^{(2-\kappa)^+}}{\lambda_+\mu^2}\right),$$

$$\text{(II)} \stackrel{def}{=} 2\mathbb{P}\left(\sup_{0\leq x\leq \frac{S_Y(1-\nu)\vee|S_Y(\mu)|}{\lambda_+ r}} L^x_{\tau(1)} > \frac{4u^0_+ + (u-u^0_+)}{\lambda_+\int_\mu^{1-\nu} x(1-x)^{\kappa-2}dx}\right),$$

$$\text{(III)} \stackrel{def}{=} \mathbb{P}\left(\int_{gr^{\theta-1}}^\infty \frac{L^y_{\tau(1)}}{y^{2-1/\kappa}}\,\mathrm{d}y > u_2 r^q\right). \tag{6.4}$$

where $g = (2\lambda_+\kappa)^{-1}$ and $u_2 = 2^{-2+1/\kappa}\lambda_+^{-1/\kappa}(u-u^0_+)$.

We know from (3.1) that (I) is of order $\mu^2 = e^{-2r^\theta}$, given the choice of μ. On the other hand, for r big enough, $\lambda_+\int_\mu^{1-\nu} x(1-x)^{\kappa-2}dx$ approaches $4u^0_+$ in which case $(4u^0_+ + (u-u^0_+))/4u^0_+$ is greater than 1, implying that (II) decays exponentially fast to zero, by virtue of (3.2).

So, keeping in mind (5.1), all we need to prove is that

$$\limsup_{r\to\infty} \frac{\log(\mathrm{III})}{\log r} \le 1-\kappa. \tag{6.5}$$

(Note that (III) gave us the right order for the lower bound.) The strategy here is akin to the one we used for bounding $\mathbb{P}_{1/2}(J_3(r,\nu)) > \delta)$ from above in the proof of (5.11).

Second use of iteration:

For all $\kappa > 1$, there exists an integer $n = n(\kappa) \ge 1$ such that $2^{n-1} < \kappa \le 2^n$. Let us suppose first that $2^{n-1} < \kappa < 2^n$. Thus, iterating n times integration by parts followed by a time change, exactly as in the proof of (5.11), leads to

$$(\mathrm{III}) \le ne^{-g_n r^{1-\theta/\kappa}} + ne^{-\hat{g}_n \log^2 r} + \mathbb{P}\left(\int_{gr^{\theta-1}}^{\infty} \frac{L^y_{\tau(1)}}{y^{2-2^n/\kappa}} dy > \frac{r^{2^n q}}{(\log r)^{2^{n+1}-2}}\right), \tag{6.6}$$

where g_n and $\hat{g}_n$ are two positive real numbers which do *not* depend on r. Note that $1-\theta/\kappa > 0$, since $\theta < 1 < \kappa$.

Now, as $\kappa/2^n < 1$, (2.9) applies, and thus the last probability is less than or equal to the probability that a stable variable of index $\kappa/2^n$ be greater than $r^{2^n q}(\log r)^{2-2^{n+1}}$. And (6.5) then follows.

For $\kappa = 2^n$, in which case stable variables are of no help, all one needs to prove, given the previous reasoning, is

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}\left(\int_{gr^{\theta-1}}^{\infty} \frac{L^y_{\tau(1)}}{y}\, dy > \frac{r^{\kappa-1}}{(\log r)^{2\kappa-2}}\right) \le 1-\kappa.$$

With the help of (2.10), the last probability is, for large r,

$$\le \mathbb{P}\left(\mathcal{C}_1 > \frac{r^{\kappa-1}}{(\log r)^{2\kappa-1}}\right) + \mathbb{P}\left(\int_0^{gr^{\theta-1}} \frac{L^y_{\tau(1)}-1}{y}\, dy < -\frac{r^{\kappa-1}}{2(\log r)^{2\kappa-2}}\right), \tag{6.7}$$

with $\mathcal{C}_1$ denoting a completely asymmetric Cauchy variable of index 1. We know that

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}\left(\mathcal{C}_1 > \frac{r^{\kappa-1}}{(\log r)^{2\kappa-1}}\right) \le 1-\kappa.$$

Hence, we are to prove the same result for the second probability in (6.7). To this end, Itô's formula for $\log u \times (1 - L^u_{\tau(1)})$ reads

$$\log(gr^{\theta-1}) \times (1 - L^{gr^{\theta-1}}_{\tau(1)}) = \int_0^{gr^{\theta-1}} \frac{1-L^u_{\tau(1)}}{u}\, du - \ 2\ \int_0^{gr^{\theta-1}} \log u \sqrt{L^u_{\tau(1)}}\, d\gamma(u),$$

with probability one, where we have used the fact that

$$\lim_{u\to 0} \frac{1-L^u_{\tau(1)}}{u^\psi}\, u^\psi \log u = 0, \quad \mathbb{P}-a.s.$$

for any $\psi > 1/2$. As a result, the second probability involved in (6.7) is, for r large enough,

$$\leq \mathbb{P}\left(L_{\tau(1)}^{gr^{\theta-1}} > \frac{r^{\kappa-1}}{(\log r)^{2\kappa}} \right) + \mathbb{P}\left(\int_0^{gr^{\theta-1}} (\log u)^2 \, L_{\tau(1)}^u \, du > \frac{r^{2\kappa-2}}{(\log r)^{4\kappa-2}} \right) + 2\,\mathrm{e}^{-\log^2 r}.$$

As is easily verified, the first probability above decays exponentially fast to zero as r approaches infinity. Now Chebychev's inequality together with the fact that $\int_0^x \log^2 u du = x^2 \log^2 x - 2x \log x + 2x$ implies that the second probability is $o(r^{1-\kappa})$. So (6.5) follows, matching the claim.

6.2 The case $Y(0) = 0$

Let $T_{1/2} = \inf\{s : Y(s) = 1/2\}$. We shall need the following result:

Lemma 6.1 *For all $n \geq 0$, we have that*

$$\mathbb{E}_0(T_{1/2}^n) < \infty.$$

We postpone the proof of this Lemma 6.1 to the end of the subsection, first showing how it will be applied. Keeping the same notation as before, we have:

The upper bound: Let $v > (1+\kappa)/(\kappa(\kappa-1))$. For $0 < \epsilon < 1$, we have

$$\begin{aligned}
\mathbb{P}_0(\Upsilon(r) > vr) &= \mathbb{P}_0\left(\int_0^r \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr \right) \leq \mathbb{P}_0\left(\int_0^{T_{1/2}+r} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr \right), \\
&\leq \mathbb{P}_0\left(\int_0^{T_{1/2}} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > \epsilon vr \right) + \mathbb{P}_0\left(\int_{T_{1/2}}^{T_{1/2}+r} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > (1-\epsilon)vr \right), \\
&\leq \mathbb{P}_0(T_{1/2} > \epsilon vr/2) + \mathbb{P}_{1/2}\left(\int_0^r \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > (1-\epsilon)vr \right), \qquad (6.8)
\end{aligned}$$

where we have used the fact that $T_{1/2}$ is $\mathbb{P}_0$-almost surely finite (provided by Lemma 6.1) together with the strong Markov property.

On the other hand, by the same reasoning we have:

The lower bound: For all $v > 0$ and all ϵ such that $0 < \epsilon < 1$,

$$\begin{aligned}
\mathbb{P}_0\left(\int_0^r \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr \right) &\geq \mathbb{P}_0\left(\int_{T_{1/2}}^r \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr;\ T_{1/2} \leq (1-\epsilon)r \right), \\
&\geq \mathbb{P}_0\left(\int_{T_{1/2}}^{T_{1/2}+\epsilon r} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr;\ T_{1/2} \leq (1-\epsilon)r \right), \\
&\geq \mathbb{P}_{1/2}\left(\int_0^{\epsilon r} \frac{Y(u)}{(1-Y(u))^2}\,\mathrm{d}u > vr \right) - \mathbb{P}_0(T_{1/2} > (1-\epsilon)r). \qquad (6.9)
\end{aligned}$$

Now, Markov's inequality in conjunction with Lemma 6.1 tells us that for all $w > 0$,

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}_0(T_{1/2} > wr) = -\infty,$$

since this limsup is less than or equal to $-n$ for any integer n; we are done by sending n to infinity.

Accordingly, choosing ϵ so small that $(1-\epsilon)w > (1+\kappa)/(\kappa(\kappa-1))$, since we have proved that (4.7) and (4.8) hold true when $Y(0) = 1/2$, (6.8) and (6.9) deliver (4.7) and (4.8), as desired.

Now we turn to Lemma 6.1. For completeness, we actually give two proofs, for there are two different ways of writing the Laplace transform of $T_{1/2}$, starting from 0. In the first proof, we exploit the fact that 0 is an entrance boundary. We begin with the Laplace transform of the first exit time of the interval $[l, 1/2]$ starting from $l < x < 1/2$, find its moments and then first send l then x to zero. In the second approach, we express the Laplace transform in terms of the hypergeometric function, then use results for special functions. Each has its advantages: while the first proof provides the finiteness of the moments by induction, the second proof, though technically much heavier, gives an explicit formula for the moments of $T_{1/2}$.

First proof of Lemma 6.1: Recall that 0 is an entrance boundary, unattainable from]0,1[. Hence, we may write

$$\mathbb{E}_0(T_{1/2}^n) = \lim_{x\to 0}\lim_{l\to 0}\mathbb{E}_x((T_l \wedge T_{1/2})^n), \tag{6.10}$$

for all $n \geq 0$ and $l < x < 1/2$. It is well-known that the Laplace transform of $T_l \wedge T_{1/2}$ (=inf$(T_l, T_{1/2})$),

$$u^l(x) \stackrel{def}{=} \mathbb{E}_x\left(\exp(-\lambda T_l \wedge T_{1/2})\right), \quad \lambda > 0,$$

satisfies $\mathcal{L}_Y u^l(x) = \lambda u^l(x)$, $l < x < 1/2$ where $\mathcal{L}_Y$ is the infinitesimal generator of Y. See for instance [21] pages 196-197. Setting

$$u_n^l(x) \stackrel{def}{=} \mathbb{E}_x((T_l \wedge T_{1/2})^n),$$

we have $u_0^l \equiv 1$ and

$$\mathcal{L}_Y u_n^l(x) = -n u_{n-1}^l(x), \quad l < x < 1/2,$$

with the boundary conditions $u_n^l(l) = u_n^l(1/2) = 0$, for all $n > 0$. Making $g \equiv n u_{n-1}^l$ in display (3.11), p 197, [21], leads to

$$\begin{aligned}\frac{2}{n}u_n^l(x) = {} & \frac{S_Y(l,x)}{S_Y(l,1/2)}\int_x^{1/2} S_Y(t,1/2)(1-t)^\kappa u_{n-1}^l(t)\,\mathrm{d}t \\ & + S_Y(x,1/2)\int_l^x \frac{S_Y(l,t)}{S_Y(l,1/2)}(1-t)^\kappa u_{n-1}^l(t)\,\mathrm{d}t,\end{aligned}$$

for all $l < x < 1/2$, $S_Y(a,b)$ denoting $S_Y(b) - S_Y(a)$, for $0 < a, b < 1$. Recall from (5.1) the definition of S_Y.

We would like to show that $u_n(0) \stackrel{def}{=} \mathbb{E}_0(T_{1/2}^n) < \infty$. We shall do so by induction. Suppose that $u_{n-1}(0) < \infty$, for $n > 1$. (This trivially holds for $n = 1$.)

Noting that $u_{n-1}^l(t) \leq u_{n-1}(0)$, for all $l < t < 1/2$, the second term in the sum above is less than or equal to $S_Y(x,1/2) \cdot u_{n-1}(0) \cdot \int_0^x (1-t)^\kappa\,\mathrm{d}t$; this is finite, independent of l and tends to zero as x goes to 0.

Moreover, at fixed x, $S_Y(l,x)/S_Y(l,1/2)$ approaches 1 as l tends to 0 since in this case $S_Y(l)$ is of order $\log l$. Therefore, sending first l to 0, at fixed x, then x to 0, we have

$$\mathbb{E}_0(T_{1/2}^n) = \lim_{x\to 0} \mathbb{E}_x(T_{1/2}^n) = \lim_{x\to 0}\lim_{l\to 0} u_n^l(x) = \frac{n}{2}\int_0^{1/2} S_Y(t,1/2)(1-t)^\kappa u_{n-1}(t)\,\mathrm{d}t < \infty,$$

by the monotone convergence theorem. We have proved that $u_n(0) < \infty$, as desired.

Second proof of Lemma 6.1: Here we write the Laplace transform of $T_{1/2}$ in a different way. Let $F(a,b,c,x)$ be the hypergeometric function. (See e.g. [1].) The function F solves the following Gaussian differential equation:

$$x(1-x)y^{''}(x) + (c-(a+b+1)x)y^{'}(x) = ab\; y(x).$$

In the light of (2.11) this can be rewritten as

$$\mathcal{L}_Y y(x) = 2ab\; y(x),$$

for $c = 1$ and $a, b > 0$ such that $a + b = 1 + \kappa$. Therefore $(F(a,b,1,Y(t))e^{-2abt})_{t\geq 0}$ is a local martingale. We apply the optional stopping theorem, getting

$$\mathbb{E}_0(e^{-2\theta T_{1/2}}) = \frac{F(a,b,1,0)}{F(a,b,1,1/2)} = \frac{1}{G(\theta)},$$

where $\theta = \theta(a) \stackrel{def}{=} ab > 0$ and

$$G(\theta) \stackrel{def}{=} F(a,b,1,1/2) = \sum_{n\geq 0} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)(n!)^2}\,\frac{1}{2^n}.$$

It follows that

$$\mathbb{E}_0(T_{1/2}^n) = (-1/2)^n \partial_\theta^n (1/G(\theta)|_{\theta=0}.$$

We note that sending for instance a to 0 and b to $1+\kappa$ sends θ to 0.

We begin by rewriting $G(\theta)$ as:

$$G(\theta) = 1 + \sum_{n\geq 1} P_n(\theta)\frac{1}{2^n(n!)^2},$$

where

$$P_n(\theta) = a(a+1)...(a+n-1)b(b+1)...(b+n-1) = \prod_{0\leq i\leq n-1} (\theta + (1+\kappa)i + i^2).$$

We shall show that $G^{'}(0) = 2\,\mathbb{E}_0(T_{1/2}) < \infty$. For higher order derivatives, the proof follows the same pattern (though with heavier expressions!).

To this end, we compute the logarithmic derivative of $P_n(\theta)$:

$$P_n^{'}(\theta) = P_n(\theta) \sum_{0\leq i\leq n-1} \frac{1}{\theta + (1+\kappa)i + i^2}.$$

It follows that for all $\theta < \delta$, with $\delta > 0$ fixed, since both $P_n(\theta)$ and $P_n(\theta)/\theta$ are increasing functions, $P_n^{'}(\theta)/2^n(n!)^2$ is bounded from above by $P_n(\delta)(1/\delta + \pi^2/6)/2^n(n!)^2$, which is summable. By Lebesgue's dominated convergence theorem followed by the monotone convergence theorem, we arrive at:

$$\begin{aligned} 2\,\mathbb{E}_0(T_{1/2}) = G^{'}(0) &= \sum_{n\geq 1} P_n^{'}(0)\,\frac{1}{2^n(n!)^2}, \\ &= \sum_{n\geq 1} \frac{\Gamma(1+n+\kappa)}{\Gamma(2+\kappa)}\,\frac{1}{2^n n\, n!} < \infty, \end{aligned}$$

as announced. The next task is to provide the

7 End of the proofs of Theorems 1.2 and 1.1.

Having proved (4.7) and (4.8), we are done with Theorem 4.1, and hence with Theorem 1.2, so long as we prove (4.9). This is the aim of the following subsection.

7.1 Proof of (4.9)

There remains to prove:

$$\limsup_{r\to\infty} \frac{1}{\log r} \log \mathbb{P}(I_1(\infty) > wr) \leq 1-\kappa,$$

for all $w > 0$. The strategy is akin to the one we used in the previous section for the tail estimates of $T_{1/2}$; we need only check:

Lemma 7.1 *For any α with $1 < \alpha < \kappa$,*

$$\mathbb{E}(I_1^\alpha(\infty)) < \infty. \tag{7.1}$$

Proof of Lemma 7.1: We go back to the identity in law provided by (4.1) with $r = \infty$, which makes sense according to (4.2), and begin with conditioning upon $\{W(y)\}_{y\in\mathbb{R}}$, and ξ, so that the only randomness in the right-hand side of (4.1) comes from the 0-dimensional squared Bessel process Z. We write $\mathbb{E}^{W,\xi}(\cdot) \stackrel{\text{def}}{=} \mathbb{E}(\ \cdot \mid \{W(y)\}_y, \xi)$ for brevity. Hölder's inequality, for any $\alpha \in (1,\kappa)$ tells us that

$$\mathbb{E}^{W,\xi}\left(I_1^\alpha(\infty)\right) \leq S^\alpha(\infty)\xi^\alpha \left(\int_{-\infty}^0 \mathrm{e}^{-\beta W(y)} g^\beta(y)\,\mathrm{d}y\right)^{\alpha/\beta} \mathbb{E}^{W,\xi}\left(\int_{-\infty}^0 g^{-\alpha}(y)\, Z^\alpha\left(\frac{|S(y)|}{S(\infty)\,\xi}\right)\mathrm{d}y\right),$$

where $g(y) \stackrel{\text{def}}{=} \mathrm{e}^{W(y)/\alpha}(|S(y)|^{\alpha-1} + S^{\alpha-1}(\infty)\xi^{\alpha-1})^{1/(\alpha\beta)}$, and $\beta > 1$ is such that $\beta^{-1} + \alpha^{-1} = 1$.

Since Z is a 0-dimensional squared Bessel process starting from 1, we can estimate its moments via its semi-group, see [31] page 441. Indeed, for any $b > 0$,

$$\mathbb{E}(Z^b(t)) = \frac{1}{2t}\,e^{-1/2t}\int_0^\infty x^{b-1/2}e^{-x/2t} I_1(\sqrt{x}/t)\,\mathrm{d}x,$$

where I_1 is the modified Bessel function of index 1, see e.g. [31] p. 549. Plugging the expression for I_1 into the above integral and using a Fubini-Tonelli argument followed by the change of variables $y = x/(2t)$, we have

$$t^{1-b}\,\mathbb{E}(Z^b(t)) = 2^b e^{-1/2t} \sum_{n\geq 0} \frac{\Gamma(n+b+1)}{n!(n+1)!} \left(\frac{2}{t}\right)^n .$$

It is then easily checked that $\limsup_{t\to\infty} t^{1-b}\,\mathbb{E}(Z^b(t)) < \infty$. This implies that there exists $h_1(b)$ such that

$$\mathbb{E}(Z^b(t)) \leq h_1(b)\,(1+t^{b-1}), \qquad t \geq 0.$$

As a consequence, taking $b = \alpha$, we obtain

$$\mathbb{E}^{W,\xi}\left(\int_{-\infty}^0 g^{-\alpha}(y)\, Z^\alpha\left(\frac{|S(y)|}{S(\infty)\,\xi}\right) \mathrm{d}y\right) \leq h_1(\alpha) \int_{-\infty}^0 g^{-\alpha}(y) \left(1 + \frac{|S(y)|^{\alpha-1}}{S^{\alpha-1}(\infty)\,\xi^{\alpha-1}}\right) \mathrm{d}y.$$

$$\mathbb{E}^{W,\xi}\left(I_1^\alpha(\infty)\right) \leq h_1(\alpha)\, S(\infty)\xi \left(\int_{-\infty}^0 \mathrm{e}^{-W(y)}(|S(y)|^{\alpha-1} + S^{\alpha-1}(\infty)\xi^{\alpha-1})^{1/\alpha}\,\mathrm{d}y\right)^\alpha .$$

Making use of (3.8), this leads to

$$\begin{aligned}
\mathbb{E}^{W,\xi}\left(I_1^\alpha(\infty)\right) &\leq h_2(\alpha)\, S(\infty)\xi \left(\int_{-\infty}^0 \mathrm{e}^{-W(y)}(|S(y)|^{1-1/\alpha} + S^{1-1/\alpha}(\infty)\xi^{1-1/\alpha})\,\mathrm{d}y\right)^\alpha \\
&\leq h_3(\alpha)\, S(\infty)\xi \left(\int_{-\infty}^0 \mathrm{e}^{-W(y)}|S(y)|^{1-1/\alpha}\,\mathrm{d}y\right)^\alpha \\
&\qquad + h_3(\alpha)\, S^\alpha(\infty)\xi^\alpha \left(\int_{-\infty}^0 \mathrm{e}^{-W(y)}\,\mathrm{d}y\right)^\alpha ,
\end{aligned}$$

with $h_2(\alpha) = h_1(\alpha) d_1^{1/\beta}(\alpha-1)$ and $h_3(\alpha) = h_2(\alpha) d_1(\alpha)$. We now take the expectation on both sides. First, since ξ is exponential of mean 2, it has finite moments of all orders. On the other hand, thanks to (2.6), $\mathbb{E}[S^b(\infty)] < \infty$ whenever $b < \kappa$. Moreover, for the same reason as before, since $\alpha < \kappa$, we have that $\mathbb{E}[\int_{-\infty}^0 \mathrm{e}^{-W(y)}\,\mathrm{d}y]^\alpha < \infty$. Accordingly, for any $\alpha \in (1,\kappa)$, we have:

$$\mathbb{E}\left(I_1^\alpha(\infty)\right) \leq h_4(\alpha,\kappa)\,\mathbb{E}\left(\int_{-\infty}^0 \mathrm{e}^{-W(y)}|S(y)|^{1-1/\alpha}\,\mathrm{d}y\right)^\alpha + h_5(\alpha,\kappa).$$

It remains for us to handle the expectation term on the right-hand side. By Lamperti's representation (2.7), we have $\int_{-\infty}^0 \mathrm{e}^{-W(y)}|S(y)|^{1-1/\alpha}\,\mathrm{d}y \overset{\text{law}}{=} 16\int_0^\infty t^{1-1/\alpha} R^{-4}(t)\,\mathrm{d}t$, where R is a Bessel process of dimension $(2+2\kappa)$ starting from $R(0) = 2$.

By Lemma 3.3, this yields $h_6(\alpha) \overset{\text{def}}{=} \mathbb{E}(\int_{-\infty}^0 \mathrm{e}^{-W(y)}|S(y)|^{1-1/\alpha}\,\mathrm{d}y)^\alpha < \infty$. As a consequence, for any $\alpha \in (1,\kappa)$,

$$\mathbb{E}\left(I_1^\alpha(\infty)\right) \leq h_4(\alpha,\kappa)\, h_6(\alpha) + h_5(\alpha,\kappa),$$

finishing the proof of Lemma 7.1. So (4.9) is proved.

We are done with the proof of Theorem 1.2. It remains to see how the '*natural duality*' between H and X enables us to translate Theorem 1.2 into Theorem 1.1. The strategy of the proof is akin to that adopted in [13] for the RWRE case.

7.2 End of proof of Theorem 1.1

We begin with:

The upper bound. Clearly, it suffices to show that for any $v \in (0, v_\kappa)$,

$$\limsup_{t\to\infty} \frac{\log \mathbb{P}(X(t) < vt)}{\log t} \leq 1 - \kappa. \tag{7.2}$$

Let $\epsilon > 0$ be given. If $X(t) < vt$, then X either stays below the level $(v+\epsilon)t$ during $[0,t]$, or hits $(v+\epsilon)t$ at time $H((v+\epsilon)t) \leq t$ and then comes below vt before time t. Accordingly,

$$\begin{aligned} &\mathbb{P}(X(t) < vt) \\ \leq\ &\mathbb{P}(H((v+\epsilon)t) > t) + \mathbb{P}(H((v+\epsilon)t) \leq t; \inf_{s \geq H((v+\epsilon)t)} X(s) < vt). \end{aligned} \tag{7.3}$$

Having in mind the definition of the annealed probability $\mathbb{P}$, the second term on the right-hand side is less than or equal to

$$E_Q\left(P^W\left(\inf_{s\geq H((v+\epsilon)t)} X(s) - X(H((v+\epsilon)t)) < -\epsilon t\right)\right). \tag{7.4}$$

Let Θ be the shift operator, defined by

$$\Theta_x W(y) = W(x+y) - W(x).$$

By virtue of the strong Markov property and the invariance of $\mathbb{P}$ under the action of the group $\{\Theta_x,\ x \in \mathbb{R}\}$, the quantity (7.4) equals

$$\begin{aligned} &E_Q\left(P^W_{(v+\epsilon)t}\left(\inf_{s\geq 0} X(s) - (v+\epsilon)t < -\epsilon t\right)\right) \\ &= E_Q\left(P^{\Theta_{(v+\epsilon)t}W}\left(\inf_{s\geq 0} X(s) < -\epsilon t\right)\right) \\ &= \mathbb{P}\left(\sup_{s\geq 0}\,(-X(s)) \ > \epsilon t\right). \end{aligned}$$

Now, thanks to [22], the last probability approaches zero exponentially fast as t goes to infinity. Accordingly, taking logarithm of (7.3), dividing by $\log r$ then taking the $\limsup$, and using (1.6), since ϵ is arbitrary, we have the upper bound of $1-\kappa$.

The lower bound. Since G is open and separated from v_κ, it suffices to establish the lower bound for $G = (v - 2\epsilon, v)$, where $0 < 2\epsilon < v < v_\kappa$. We set

$$\mathcal{L}_y = \sup_{t\geq H(y)} (y - X(t)),$$

and observe that the event $\{X(t)/t \in (v-2\epsilon, v)\}$ contains the event

$$\begin{aligned} &\left\{\frac{(v-2\epsilon)}{v_\kappa} t < H((v-\epsilon)t) < t\,; H(vt) > t;\ \mathcal{L}_{(v-\epsilon)t} < \epsilon t\right\} \\ &\stackrel{def}{=} A_t \cap B_t \cap C_t. \end{aligned}$$

Clearly,

$$
\begin{aligned}
\mathbb{P}\left(\frac{X(t)}{t} \in (v-2\epsilon, v)\right) &\geq \mathbb{P}(A_t \cap B_t \cap C_t), \\
&\geq \mathbb{P}(B_t|A_t)\mathbb{P}(A_t) - \mathbb{P}(C_t^c). \qquad (7.5)
\end{aligned}
$$

Now, since $\kappa > 1$, we know from [23] that $H(r)/r$ approaches $4/(\kappa-1) = v_\kappa^{-1}$, $\mathbb{P}$-almost surely, as r tends to infinity. Thus, as $v < v_\kappa$,

$$
\lim_{t\to\infty} \mathbb{P}(A_t) = 1. \qquad (7.6)
$$

On the other hand, once again the strong Markov property together with the invariance of $\mathbb{P}$ under $\{\Theta_x,\ x \in \mathbb{R}\}$ and [22] imply that

$$
\mathbb{P}(C_t^c) = \mathbb{P}(\mathcal{L}_{(v-\epsilon)t} > \epsilon t) = \mathbb{P}\left(\inf_{s\geq 0} X(s) < -\epsilon t\right) \qquad (7.7)
$$

is exponentially small as $t \to \infty$.

Lastly, since $H((v-\epsilon)t)$ does not depend on $\{W(x); x \geq (v-\epsilon)t\}$, it follows by stationarity that

$$
\begin{aligned}
\mathbb{P}(B_t|A_t) &\geq \mathbb{P}\left(H(vt) - H((v-\epsilon)t) > (1 - \frac{v-2\epsilon}{v_\kappa})t \ \mid A_t\right), \\
&= \mathbb{P}\left(H(\epsilon t) > \left(1 - \frac{v-2\epsilon}{v_\kappa}\right)t\right). \qquad (7.8)
\end{aligned}
$$

Putting (7.5), (7.6), (7.7), (7.8) and (1.7) together completes the proof of the lower bound in Theorem 1.1. □

Although we had come up with the iteration scheme as a way of avoiding the technical difficulty associated to a Sturm-Liouville approach, upon the prodding of the referee we were in fact able to push through that method as well. So for completeness, we include this approach in the next section.

8 A Sturm-Liouville alternative to the iteration scheme

In (5.17) and (6.4), the iteration scheme enabled us to prove that

$$
\limsup_{r\to\infty} \frac{1}{\log r}\ \log \mathbb{P}\left(\int_\epsilon^\infty \frac{L_{\tau_1}^y}{y^{1+\gamma}}\,\mathrm{d}y > r^\gamma\right) = -\infty, \qquad (8.1)
$$

for $\epsilon = \epsilon(r) = r^{\theta-1} \to 0$, $(0 < \theta < 1)$, with $\gamma > 0$ equal to $1 + 1/\kappa$ in (5.17) and to $q = 1 - 1/\kappa$ in (6.4).

An alternative way of estimating the tails of $\int_\epsilon^\infty L_{\tau_1}^y / y^{1+\gamma}\,\mathrm{d}y$ is to study its Laplace transform. Thanks to a result of Pitman and Yor [28], this reduces to solving a Sturm-Liouville equation, as we will see in this section.

From [28], we get that, for all $\lambda > 0$,

$$\mathbb{E}\left(\exp\left(-\lambda\int_0^\infty \frac{L^y_{\tau_1}}{y^{1+\gamma}}\,1_{y\geq\epsilon}\ \mathrm{d}y\right)\right) = e^{\phi'_\lambda(0^+)/2}, \tag{8.2}$$

with $\phi'_\lambda(0^+)/2$ denoting the right-derivative of ϕ_λ at 0, where ϕ_λ is the unique convex, decreasing, nonnegative solution of the following Sturm-Liouville equation with $\phi_\lambda(0) = 1$:

$$\Phi''_\lambda(x) - \frac{2\lambda}{x^{1+\gamma}}\,1_{x\geq\epsilon}\ \Phi_\lambda(x) = 0.$$

Note that one should a priori multiply $e^{\phi'_\lambda(0^+)/2}$ by $\phi_\lambda(\infty)^0$ in (8.2). The convention $0^0 = 1$ allows us to omit this factor.

Solving the Sturm-Liouville equation amounts to solving the following Riccati's differential equation with $y = \Phi'_\lambda/\Phi_\lambda$:

$$y'(x) + y^2(x) = 2\lambda\ x^{-1-\gamma}, \quad x \geq \epsilon.$$

We find from [39], p 88-89, that this is soluble in finite terms only when $(1-\gamma)/2$ is the inverse of an odd integer, that is when $\kappa = n + 1/2$, for $n \geq 1$.

For arbitrary $\kappa > 1$, and $\lambda > 0$, the general solution of our Sturm-Liouville equation reads:

$$\Phi_\lambda(x) = \sqrt{x}\ \mathcal{C}_\kappa(i\kappa\sqrt{8\lambda}x^{(1-\gamma)/2}), \quad x \geq \epsilon,$$

with $\mathcal{C}_\kappa$ a cylindrincal function of index κ; see [39], pages 82-83.

Now Φ'_λ is constant on the interval $[0, \epsilon]$; a few lines of computation give that for $\epsilon > 0$,

$$\Phi'_\lambda(0^+) = \Phi'_\lambda(\epsilon) = i\ \sqrt{2\lambda}\epsilon^{-\gamma/2}\ \left(\mathcal{C}_{\kappa-1}(i\kappa\sqrt{8\lambda}\epsilon^{1/(2\kappa)})\ 1_{\gamma=q}\ +\ \mathcal{C}_{\kappa+1}(i\kappa\sqrt{8\lambda}\epsilon^{-1/(2\kappa)})\ 1_{\gamma=1+1/\kappa}\ \right),$$

with 1_A denoting the indicator function of A.

From the analyticity of $\mathcal{C}_\kappa$ one gets that, as a function of $\lambda > 0$, $\Phi'_\lambda(\epsilon)$ is analytic, thus for $\lambda > 0$ small enough (depending on ϵ or equivalently on r), one could write:

$$2\ \log\ \mathbb{E}\left(\exp\left(\lambda\int_\epsilon^\infty \frac{L^y_{\tau_1}}{y^{1+\gamma}}\ \mathrm{d}y\right)\right) = \phi'_{-\lambda}(\epsilon) = \sqrt{2\lambda}\epsilon^{-\gamma/2}\ \mathcal{C}_{\kappa\pm1}\left(\kappa\sqrt{8\lambda}\epsilon^{\mp1/(2\kappa)}\right),$$

for *the* cylindrical function $\mathcal{C}_\kappa$ determined by the particular solution ϕ_λ.

A cylindrical function can be expressed as:

$$\mathcal{C}_\kappa(x) = a_\kappa\ J_\kappa(x) + b_\kappa\ Y_\kappa(x),$$

with a_κ and b_κ two periodic functions of κ with period one, and where J_κ and Y_κ are Bessel functions of the first and second kind respectively. From pp 622, 625 and 627 of [15] (or pp 74 and 199 of [39]) we have the asymptotic equivalents of J_κ and Y_κ at 0 and infinity: for x in the neighborhood of 0, $J_\kappa(x)$ is of order x^κ and $Y_\kappa(x)$ of order $x^{-\kappa}$ (for $\kappa > 1$). Furthermore, for x large, both $J_\kappa(x)$ and $Y_\kappa(x)$ are of order $x^{-1/2}$.

This provides all the ingredients for proving our tail estimates. Indeed, for all $u > 0$, with the previous choices of $0 < \theta < 1$, $\epsilon = r^{1-\theta}$, and for $\lambda = \lambda(r)$ chosen to go very slowly to zero as r tends to infinity, an exponential inequality together with (8.2) yields

$$\mathbb{P}\left(\int_\epsilon^\infty \frac{L_{\tau_1}^y}{y^{1+\gamma}}\ \mathrm{d}y > u\, r^\gamma\right) \leq \exp(-\lambda u r^\gamma + \frac{1}{2}\phi'_{-\lambda}(\epsilon)).$$

By virtue of the choice of λ, $\phi'_{-\lambda}(\epsilon)$ is of order $\epsilon^{-\gamma} = r^{(1-\theta)q} = o(r^\gamma)$, for $\gamma = 1 - 1/\kappa$, and $r^{(1-\theta)(\gamma+1)/4} = o(r^\gamma)$, for $\gamma = 1 + 1/\kappa$.

We have proved (8.1) for both (5.17) and (6.4).

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Marina Talet
C.M.I. Université de Provence
LATP, CNRS-UMR 6632
39, rue F. Joliot Curie
F-13453 Marseille Cedex 13
France
marina@cmi.univ-mrs.fr